14.

INTRODUCTION TO FACTORIAL DESIGNS

14.1 FACTORIAL DESIGNS

By a *factor* we mean a controllable input. We will denote factors by capital letters A, B, C… Each factor can take a certain number of *levels* or values in an experiment.

We denote the levels

A: $A_1, A_2, \ldots, A_a$
B: $B_1, B_2, \ldots, B_b$

A combination of levels, one from each factor in the experiment is called a *treatment* and denoted

$A_{i1}\, B_{i2}\, C_{i3}$

An experiment in which each possible treatment is applied is called a completely crossed design or simply a factorial design.

For example we have 3 factors A, B, C where: A takes a levels,
B takes b levels,
C takes c levels.

Then there are abc possible treatments. If treatment $A_{i1}\, B_{i2}\, C_{i3}$ is applied $n_{i1\, i2\, i3}$ times then we have a total of

$$\sum_{i_1=1}^{a} \sum_{i_2=1}^{b} \sum_{i_3=1}^{c} n_{i1\, i2\, i3}$$

treatment applications.
Rather than running the full 3-factor design we could think of running 3 one factor designs.

For this we would require only a + b + c ≤ abc treatments. This would certainly be much easier. We note, however, that if we did this we would have no way of assessing whether or not the factors interacted. Thus the importance of factorial designs in general. Typically each treatment is applied more than once. If each treatment is applied the same number of times we have a balanced design. Balance *greatly* simplifies computations and interpretations and should be arranged if possible.

Now suppose we have k factors and factor i takes m_i levels. Then we have $m_1\, m_2 \ldots m_k$ treatments and suppose treatment $(i_1, i_2, \ldots, i_k)$ is to be applied $n_{i1\ i2 \ldots ik}$ times.

Then we require $N = \sum_{i_1=1}^{m_1} \cdots \sum_{i_k}^{m_k} n_{i1 i2 \ldots ik}$

elements in our population Π so that all treatments can be applied.

The elements of Π are usually referred to as *experimental units* in this context.

We then randomly assign units to treatments. This process is called complete randomization and ensures that we have random samples from each of the $m_1\, m_2 \ldots m_k$ frequency distributions determined by the treatments.

We note that formally any inferences or conclusions we read apply only to the population Π unless the elements of Π were themselves a random sample from some larger population. In practical contexts, however, psychologists often make such an extrapolation whether justified or not.

14.2 THE 1-FACTOR DESIGN

Suppose we have 1 factor A: $A_1, \ldots, A_a$ that has a levels. This gives raise the linear model

$$\boxed{E[y] = \beta_1 x_1 + \ldots + \beta_a x_a}$$

where $x_i = 1$ if A_i is applied and is zero (0) otherwise.

If we apply $A_i n_i$ times we obtain a $N = n_1 + n_2 + \ldots + n_a$ dimensional data vector y, where the first n_1 components are from A_1, etc.

Then the full model takes the form

$$E[y] = \mathbf{X}\beta$$
$$= (x_1, \ldots, x_a)\beta$$

where x_i has 1's in the $n_1 + \ldots + n_{i-1} + 1$ *to* $n_1 + \ldots + n_i$ positions and has 0's elsewhere. The least-squares estimator of β is given by

$b = (X'X)^{-1} X'y$

$$= \left(\frac{T_1}{n_1}, \ldots, \frac{T_a}{n_a} \right)$$

$$= \left(\bar{y}_1, \ldots, \bar{y}_a \right)$$

where T_i = the total of all the observations from an application of A_i and $\bar{y}_i = \frac{T_i}{n_i}$.

Note that the psychologist has defined the model and has collected data on the system.

The least squares estimator, b of β, is the mean of the treatment, $\bar{y}$.

This gives the following ANOVA table:

Source	DF	SS	MSE	F
Model	a	$\sum_{i=1}^{a} \frac{T_i^2}{n_i}$	$\frac{\sum \frac{T_i^2}{n_i}}{a}$	$\frac{MSM}{s^2}$
Error	N-a	$y'y - \sum \frac{T_i^2}{n_i}$	s^2	
Total	N	$y'y$		

Further the F test on a and N-a degrees of freedom tests H_0: β = 0; i.e. all the means are equal and they are equal to zero. This is a hypothesis that is seldom of interest to psychologists. Typically psychologists are interested in testing if there are treatment differences, not that the treatment means are equal to zero.

14.3 TEST FOR NO TREATMENT DIFFERENCES

A more common hypothesis that we wish to test is

H_0: $\beta_1 = \beta_2 = \ldots = \beta_a$;

i.e. there are no treatment differences or equivalently no cause-effect relationship exists between A and the response variable. Note that this is a different hypothesis than that tested in the previous section.

This hypothesis corresponds to

H_0: $E[y] \in X_2 \beta_2$ given that $E[y] \in X_1 \beta_1$

where $X_2 = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$ and X_1 is the design matrix for the full model.

We obtain the following ANOVA table using the ANOVA technique previously discussed in Chapter 11.

Source	DF	MSE	F
Model X_2	1	$\left(\frac{G^2}{N}\right)$	
$\frac{Model\ X_1 - Model\ X_2}{(difference)}$	a-1	$\frac{\left(\sum_{i=1} \frac{T_i^2}{n_i} - \frac{G^2}{N}\right)}{(a-1)}$	$\left(\frac{MSD}{s^2}\right)$
Error (Model X_1)	N-a	s^2	
Total	N		

where G is the grand total of all the observations.

We test H_0 by comparing F with the F(a-1, N-a) distribution.

This table is usually presented as follows in introductory textbooks on statistics and computer program output.

ANOVA (one way)	DF	MSE	F
Between Groups	a-1	MSB	$\frac{MSB}{MSW}$
Within Groups	N-a	MSW	
Total (Removing the mean)	N-1		

The difference in models (Model X_1 - Model X_2) is referred to as Between Group variation whereas the error or residual component of the model is referred to as Within Group variation.

14.4 CONTRASTS

Definition of an Orthogonal matrix

1) For any two rows, the sum of products of corresponding entries is zero.
2) For any row, the sum of the squares of the entries is 1.

Example 1 Consider the matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ -5 & 4 & 1 \end{bmatrix}$$

Multiply row 1 and row 2 of **A**, in order to verify rule 1 above.

$r_1 * r_2' = 1 * 1 + 1 * 2 + 1 * (-3)$
$= 1 + 2 - 3$
$= 0$

Continue with row 2 and row 3.

$r_2 * r_3' = 1 * (-5) + 2 * 4 + (-3) * 1$
$= -5 + 8 - 3$
$= 0$

for practice, verify the product of row 1 and 3 equals zero.

To satisfy this second requirement divide each row by the square root corresponding sum of squares of each row.

i.e. $r_1 * r_1' = 1 * 1 + 1 * 1 + 1 * 1$
$= 1 + 1 + 1$
$= 3$

thus the divisor is $\sqrt{3}$ for row 1.

For row 2 we obtain

$r_2 * r_2' = 1^2 + 2^2 + (-3)^2$
$= 1 + 4 + 9$
$= 14$

thus the divisor is $\sqrt{14}$ for row 2 and similarly
the divisor is $\sqrt{42}$ for row 3.

Thus the matrix **C** below is an orthogonal matrix

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}} \end{bmatrix}$$

For practice, check to see that **C** is an orthogonal matrix - it satisfies 1) and 2). You will soon see that SPSS does not require the divisor we have introduced. We have introduced the divisor so that our hand calculations are complete.

Now suppose **C** is orthogonal and C_1 - the first row of **C** is (1 1 … 1) with divisor $\sqrt{a}$ and we wish to make inferences about the contrasts $\alpha = C\beta$,

where our model is

$E[y] = X\beta$

Typically there is a natural ordering to these contrasts and we test sequentially.

H_0: $\alpha_a = 0$
H_0: $\alpha_{a-1} = 0$ assuming $\alpha_a = 0$
etc.

and stop when we obtain a significant result. We obtain the following ANOVA table.

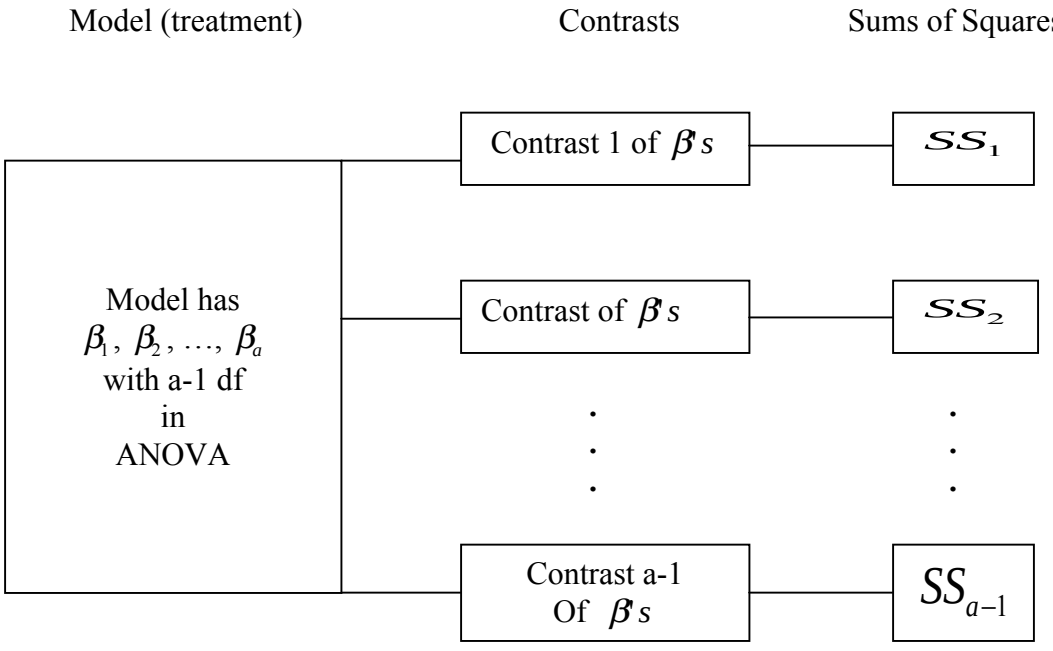
Source	DF
$\alpha_a = 0$ (contrast 1)	1
$\alpha_a - 1 = 0$ (contrast 2)	1
.	
.	
.	
Residual (Error)	N-a
Total	N

Note that the sums of squares (SS) for each contrast is na_i^2 where

$$a_i = C_i \begin{bmatrix} \frac{T_1}{n} \\ . \\ . \\ . \\ \frac{T_a}{n} \end{bmatrix}$$

$$= C_i \begin{bmatrix} \bar{y}_1 \\ . \\ . \\ . \\ \bar{y}_a \end{bmatrix}$$

Orthogonal contrasts are important since the mathematical property of orthogonality gives the set of orthogonal contrasts described in the matrix **C** mutual statistical independence. The contrasts also partition the Treatment variation into independent components that when summed give the Treatment variation. Research can be designed with a specific set of contrasts in mind or as we will see in Chapter 19 psychologists can examine treatments without a set of prescribed contrasts.



(d) Example

Gebotys and Roberts, *Canadian Journal Behavioural Science* (1987) 19, p 479, examined the public's attitude towards sentencing by giving each subject a short story describing a crime and then asking the subject to sentence the offender. The data below are based on this study. There are three treatment/crime conditions; break and enter (A_1), robbery (A_2) and manslaughter (A_3). Three subjects are randomly assigned to each treatment for a total of 9 subjects and a sentence, in months, is recorded.

A reading of the literature indicates that there may be a difference between the mean sentence of A_2 and the mean sentence of A_3 and then if the difference doesn't exist, it seems reasonable to test for a difference between the mean of A_1 and the common value (average value) for A_2 and A_3.

The data, three observations per treatment, with totals and means is given below.

A_1: 38.5,39.8,37.2	$T_1 = 115.5$	$\bar{y}_1 = 38.5$
A_2: 40.1,41.5,39.3	$T_2 = 120.9$	$\bar{y}_2 = 40.3$
A_3: 40.1,43.2,42.2	$T_3 = 125.5$	$\bar{y}_3 = 41.833$
	$G = 361.9$	

The following calculations are necessary for the construction of the ANOVA table.

$$\frac{G^2}{N} = 14552.40111$$

$$\sum_{i=1}^{3} \frac{T_i^2}{n} = 14569.10111 \qquad \sum \frac{T_i^2}{n} - \frac{G^2}{n} = 16.7 \qquad \text{y'y} = 14579.97$$

To test if sentencing is equivalent for the three types of crime we obtain the following table.

Source	DF	SS	MS	F
Mean	1	14552.40111	14552.40111	
Treatments	2	16.7	1.81	4.61
Error	6	10.87	8.35	
Total	9	14579.97111		

If we test at $\alpha = .05$, the critical value for $F(2, 6) = 5.79$ therefore we cannot reject H_0: $\beta_1 = \beta_2 = \beta_3$; that the three means are equal. The observed level of significance or p value is equal to about .06.
In other words, we have marginal evidence against the hypothesis of no treatment differences.

The experiment was designed with the following two orthogonal contrasts in mind

1) the mean of A_2 vs. the mean of A_3
2) the mean of A_1 vs. the mean of A_2 and A_3

This leads to the contrast matrix

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

and contrasts for the population

$$\alpha = C\beta = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\beta_1 + \beta_2 + \beta_3)}{\sqrt{3}} \\ \frac{(2\beta_1 - \beta_2 - \beta_3)}{\sqrt{6}} \\ \frac{(\beta_2 - \beta_3)}{\sqrt{2}} \end{bmatrix}$$

Now since $n_1 = n_2 = n_3$, let the sample size per treatment equal n. We calculate contrasts for the sample using totals or averages.

$$a_1 = \begin{pmatrix} \frac{\frac{T_1}{3} + \frac{T_2}{3} + \frac{T_3}{3}}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{T_1 + T_2 + T_3}{3\sqrt{3}} \end{pmatrix}$$

$$a_2 = \begin{pmatrix} \frac{2T_1 - T_2 - T_3}{3\sqrt{6}} \end{pmatrix}$$

$$a_3 = \begin{pmatrix} \frac{T_2 - T_3}{3\sqrt{2}} \end{pmatrix}$$

Remember that the SS for the contrast is na_i^2 where n = 3 and a_i is calculated above.

Then the relevant ANOVA table is given by

Source	DF	MS	F
Mean	1	14552.40111	
A_1 vs. A_2 and A_3	1	13.18	7.27
A_2 vs. A_3	1	3.53	1.95
Error	6	1.81	
Total	9		

Note that the SS for Treatment has been partitioned into two orthogonal components each with one degree of freedom which, when added together, give the Treatment Sums of Squares. At $\alpha = .05$ (critical value for $F(1,6) = 5.99$) we have no evidence against the null hypothesis of no difference between A_2 vs. A_3, however there is a significant difference between A_1 vs. A_2, A_3. This is the case since the overall F test is a pooling of the two contrasts. Since one contrast was significant and the other not, the pooling gave results approaching significance in the overall test. If an experiment has been designed with specific orthogonal contrasts in mind, then researchers can proceed to test these hypothesis even though the overall pooled test may be non significant, as is the case in our example.

A 95% C.I. for α_i is given by

$$a_i \pm t_{.025}(6)\ \frac{s}{\sqrt{n}}$$

where s is the square root of the MS Error value (with 6 df) and n is the number of observations per treatment (i.e. 3). Since $\alpha = .05$ the t value we look up is $\frac{\alpha}{2} = .025$ with 6 df which is equal to 2.44.

For example for a_2 we have

$$a_2 \pm \frac{\sqrt{1.81}}{\sqrt{3}}(2.44)$$
$$= a_2 \pm 1.895$$
$$= -2.09 \pm 1.895$$
$$= (-3.99, 0.195)$$

A 95% confidence interval for a_2 the difference between the average of A_2 and A_3 vs. A_1 is (-3.99, -0.195).

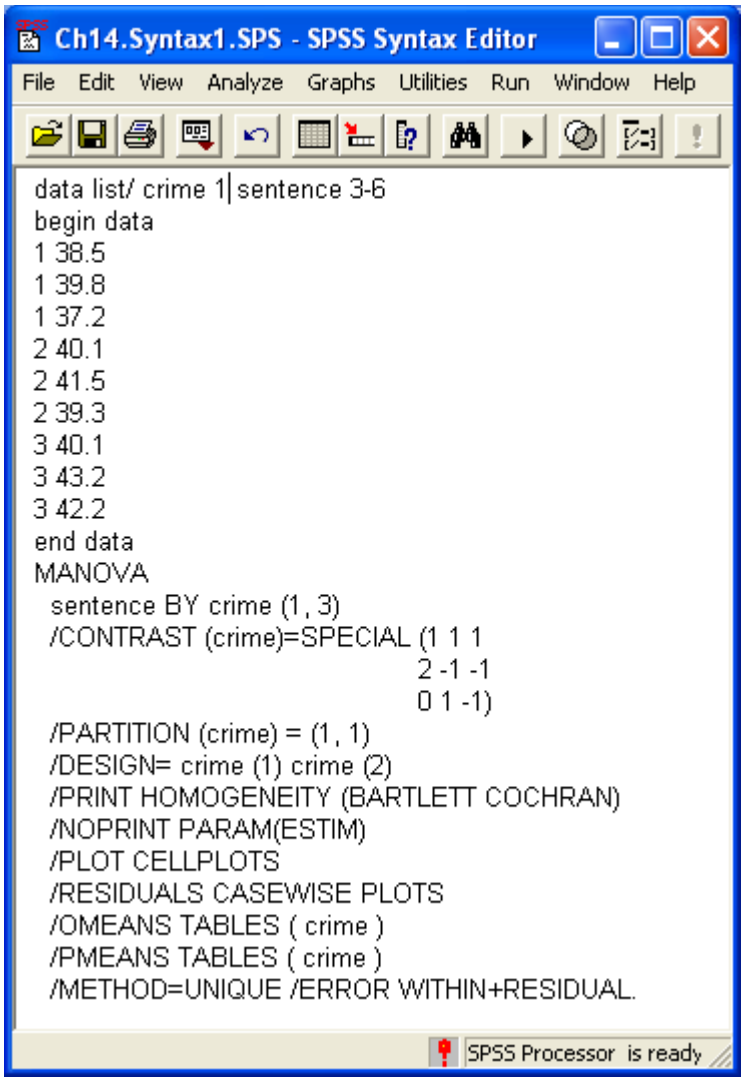
14.5 SPSS COMMANDS

The following commands will implement the above one-way ANOVA analysis using the MANOVA procedure in SPSS. The program is given below.

You have two choices regarding how to enter in your data when running a syntax program: 1) you may enter the data right into the syntax window, or 2) you may enter the data into the data editor and type the syntax separately.

If you want to enter the data directly into the syntax window, please follow these steps:

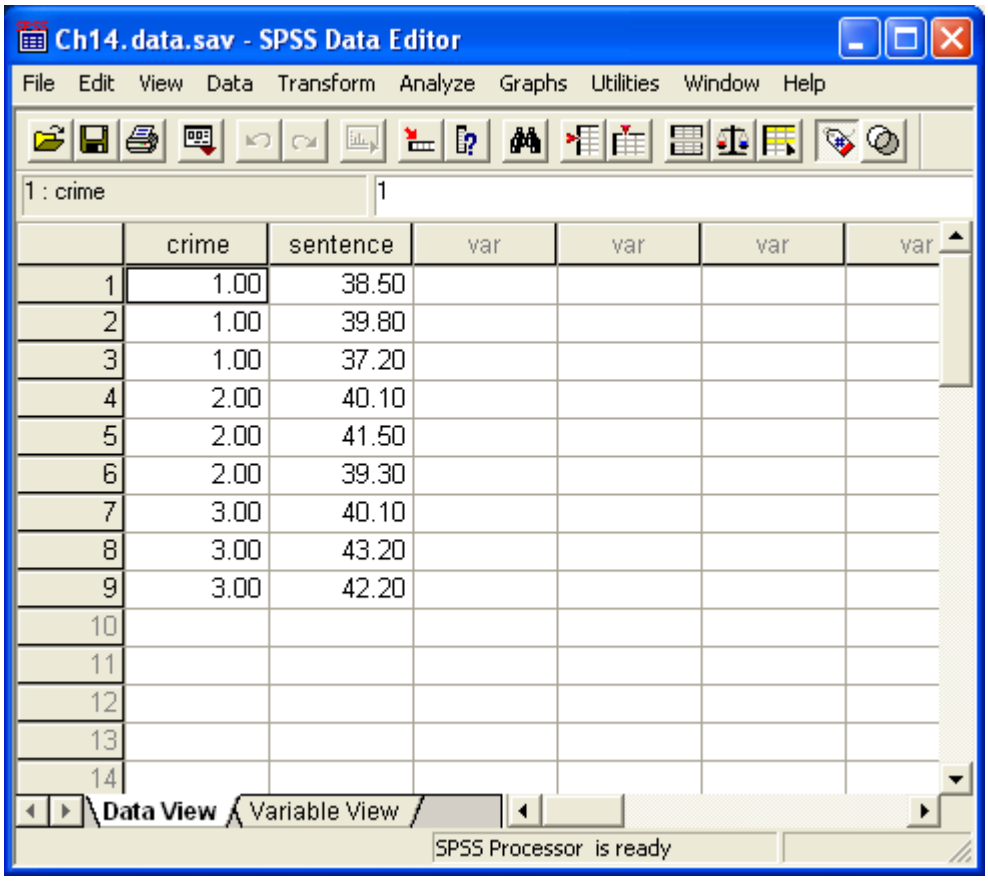
a) Click **File** on the menu bar, then **New**, followed by **Syntax**. This series of clicks will open a SPSS Syntax Editor window. Type the following commands into the Syntax Editor. Be sure to type the Syntax exactly as you see it in the picture provided below (i.e., use the appropriate case, spaces between words, etc.).



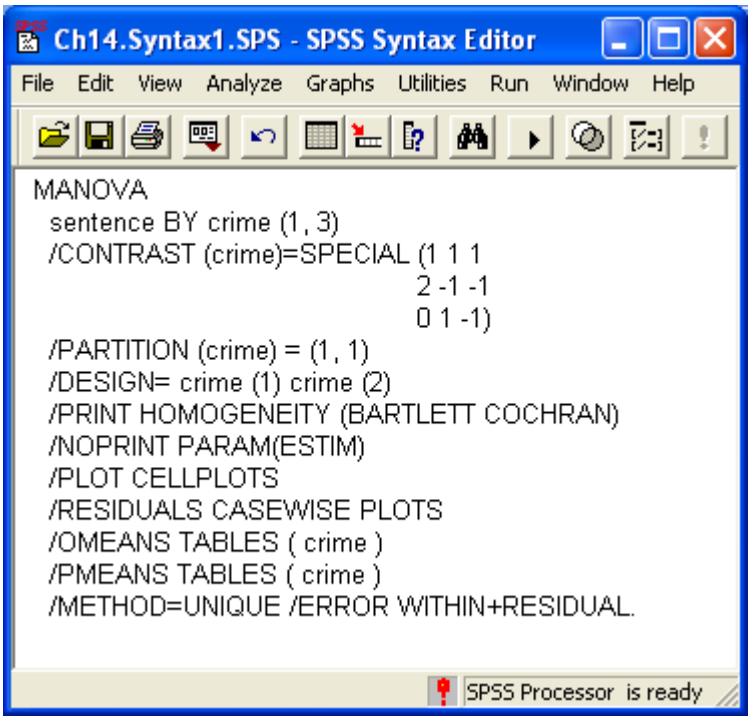
b) When you have completed typing the above commands into the Syntax window, click on **Run**, and then **All**. Your output will then be available for you to view.

If you prefer to enter you data in separately from your syntax, please follow these steps:

a) Open up SPSS and enter your data into the **Data Editor** (the main data page). Your data and a picture of the data editor have been provided below:



b) Now you can enter your commands into the Syntax Window. Click **File** on the menu bar, then **New**, followed by **Syntax**. This series of clicks will open a SPSS Syntax Editor window. Type the following commands into the Syntax Editor. Be sure to type the Syntax exactly as you see it in the picture provided below (i.e., use the appropriate case, spaces between words, etc.).



c) When you have completed typing the above commands into the Syntax window, click on **Run**, and then **All**. Your output will then be available for you to view.

The command statement is MANOVA. On the same line we have the dependent measure (y), sentence followed by the factor, crime, with three levels coded as either 1, 2, or 3, where 1 is the minimum value and 3 is the maximum value. The contrast statement is identical to the contrast matrix in form however, note that in the computer implementation it is not necessary to provide the divisor for the contrast. This is automatically calculated by the computer. The partition statement includes (1,1) two ones since there are 2 degrees of freedom for the crime factor (or two (2) orthogonal contrasts) denoted crime (1), and crime (2) in the Decision statement.

The coarse ANOVA table is listed below which tests the hypothesis

H_0: $\beta_1 = \beta_2 = \beta_3$
H_a: $\beta_1 \neq \beta_2 \neq \beta_3$

Note that the β's refer to the population means, whereas the b's refer to the sample means. Since $F(2, 6) = 4.61$, $p = .061$, we are unable to reject the null hypothesis at $\alpha = .05$.

```
Tests of Significance for SENTENCE using UNIQUE sums of squares
 Source of Variation          SS      DF        MS         F  Sig of F

 WITHIN+RESIDUAL           10.87       6      1.81
 CRIME                     16.70       2      8.35      4.61      .061

 (Model)                   16.70       2      8.35      4.61      .061
 (Total)                   27.57       8      3.45

 R-Squared =           .606
 Adjusted R-Squared =  .474
```

The means are given below for each of the three levels of the crime factor.

```
Adjusted and Estimated Means
 Variable .. SENTENCE
  CELL         Obs. Mean   Adj. Mean   Est. Mean  Raw Resid. Std.
Resid.

     1          38.500      38.500      38.500        .000        .000
     2          40.300      40.300      40.300        .000        .000
     3          41.833      41.833      41.833        .000        .000
```

The fine analysis of variance table, including the two contrasts referred to as CRIME(1) and CRIME(2) is given below. The contrast of A_1 vs. $A_3 + A_2$ is significant, $F(1, 6) = 7.27$, $p = .036$, whereas the A_2 vs. A_3 contrast is not, $F(1, 6) = 1.95$, $p = .212$. Note that the overall test given in the coarse ANOVA was *not* significant however one of the orthogonal contrasts *was* significant.

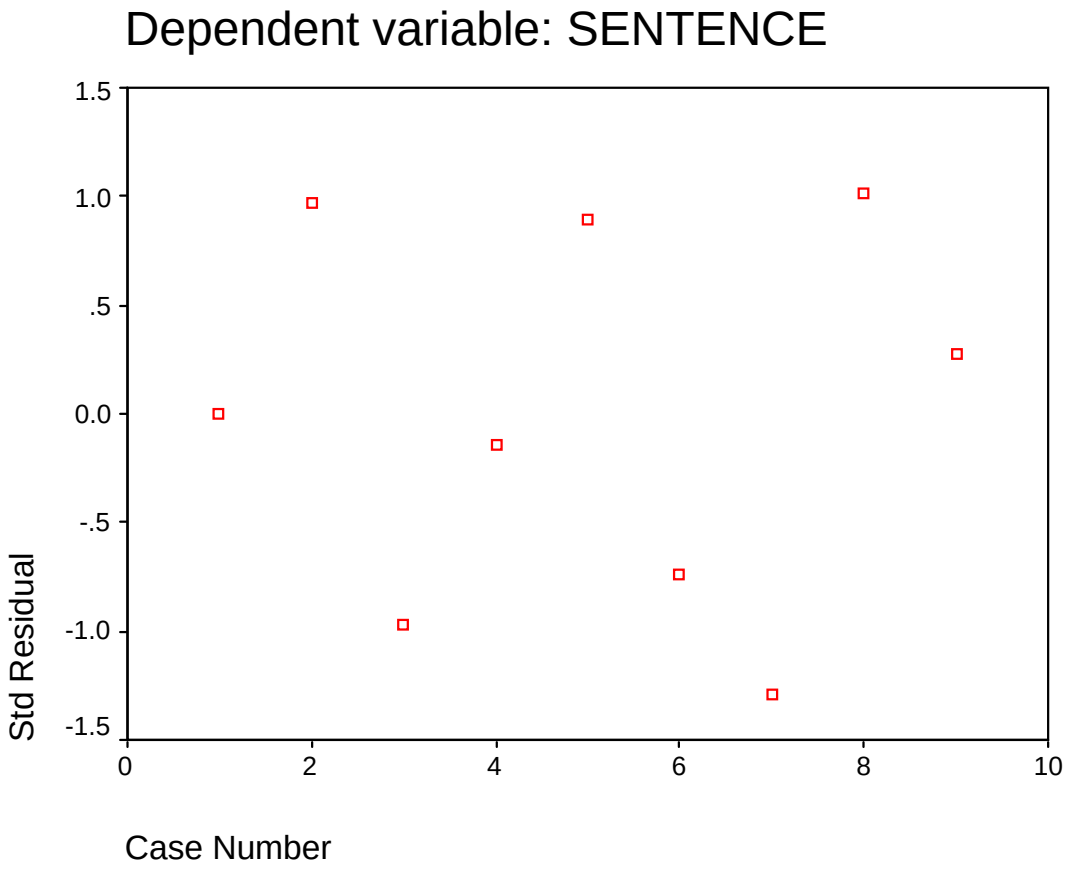
```
 Tests of Significance for SENTENCE using UNIQUE sums of squares
 Source of Variation          SS      DF        MS         F  Sig of F

 WITHIN+RESIDUAL           10.87       6      1.81
 CRIME(1)                  13.18       1     13.18      7.27      .036
 CRIME(2)                   3.53       1      3.53      1.95      .212

 (Model)                   16.70       2      8.35      4.61      .061
 (Total)                   27.57       8      3.45

 R-Squared =           .606
 Adjusted R-Squared =  .474
```

A casewise plot of residuals is given below. The standardized residual column does not display any large (greater than 3) residuals.

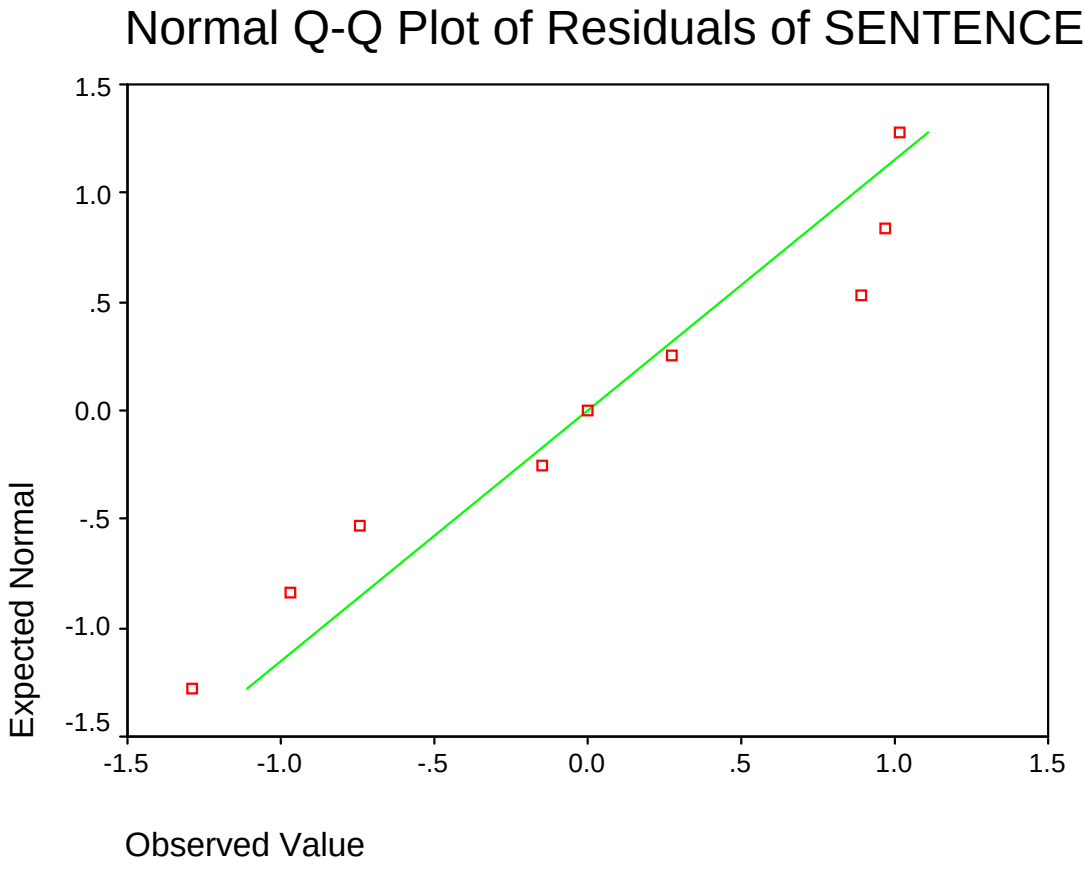
Observed and Predicted Values for Each Case
Dependent Variable.. SENTENCE

Case No.	Observed	Predicted	Raw Resid.	Std Resid.
1	38.500	38.500	.000	.000
2	39.800	38.500	1.300	.966
3	37.200	38.500	-1.300	-.966
4	40.100	40.300	-.200	-.149
5	41.500	40.300	1.200	.892
6	39.300	40.300	-1.000	-.743
7	40.100	41.833	-1.733	-1.288
8	43.200	41.833	1.367	1.016
9	42.200	41.833	.367	.272

The plot of the case number vs. standardized e_i has a band pattern that looks reasonable. Remember a linear model is being fit to the data therefore the techniques used to assess normality in this experimental design case are identical to what was learned in Part II (Chapters 7 through 13) the Linear Model.



The normal probability plot approximates a line which looks reasonable as well.



In summary, the one-way ANOVA design uses a linear model where the β's of the model represent the population means μ. The least squares estimators b are the sample means of the treatments $\bar{y}$. Contrasts of means are directly linked to the linear model. Residual analysis of the design is identical to what was learned previously in the Linear Model.

Click here for the SPSS windows method of analysis and output.

14.6 SAS COMPUTER IMPLEMENTATION

Listed below is the SAS program that would implement the one-way ANOVA analysis in section 14.4.

```
DATA OFFENCE;
INPUT CRIME SENTENCE;
CARDS;
1 3 8.5
...
3 4 2.2
PROC GLM;
CLASS CRIME;
```

```
MODEL SENTENCE = CRIME;
MEANS CRIME / BON;
CONTRAST 'A2 VS A2' CRIME 0 1 -1 ;
CONTRAST 'A 2 + A3 VS A1' CRIME 2 -1 -1 ;
OUTPUT OUT = RESIDS P = YHAT R = RESID;
PROC PLOT;
PLOT RESID*(YHAT,CRIME);
PROC UNIVARIATE PLOT NORMAL;
VAR RESID;
```

The procedure is called GLM. The CLASS command identifies the independent factor labeled CRIME in this case. The MODEL statement gives the dependent variable, SENTENCE and the independent factor CRIME. The means for the CRIME factor are requested in the MEANS statement as well as Bonferroni pair wise comparisons using the BON option (see section 14.7 for more information). Contrasts are given and labeled in the CONTRAST statement. The usual (previously discussed) residual analysis follows the CONTRAST statements.

14.7 PLANNED NON-ORTHOGONAL COMPARISONS

The *Bonferroni* method is an exact method that is applicable to a wide variety of contrasts. Psychologists often are in the position in practice of performing non-orthogonal *planned* comparisons (other non-planned comparisons are discussed in Chapter 19), including pair wise comparisons. The basis of the Bonferroni method is that if ω comparisons are to be made each with confidence

$$1 - \frac{\alpha}{\omega}$$

then the probability of making one or more Type I errors is at most α.

For pair wise comparisons of $\bar{y}_i$ vs. $\bar{y}_j$ with equal n for a treatments.

Calculate

$$\boxed{\text{Bonf} = t\,\frac{\alpha}{2\omega}\,s\sqrt{\frac{2}{n}}}$$

where,

ω = number of pair wise comparisons

If all pairwise comparisons are to be examined then

$$\omega = \left(\frac{a}{2}\right) = \left(\frac{a\,(a-1)}{2}\right)$$

where

$$s = \sqrt{MSE}$$

If the mean difference is greater than Bonf then the means differ.

Example

In the previous example we have 3 means, a = 3, the number of total pair wise comparisons is $\omega = \frac{3(3-1)}{2} = 3$

then $2\omega = 6$

where $s = \sqrt{1.81}$
$= 1.345$

and DFE = 6

n = 3 since there are 3 observations per treatment.

if we use $\alpha = .05$ then

$$t\left(\frac{a}{2\varpi}\right) = t\left(\frac{0.05}{6}\right) = t_{0.0083}$$

using tables, $t_{0.0083} \approx 3.372$ with 6df.

$$Bonf = t_{0.0083}\, s \sqrt{\frac{2}{n}}$$
$$= 3.372 \sqrt{1.81} \sqrt{\frac{2}{3}}$$
$$= 3.703$$

In other words, a pair of sample means must differ by more than 3.703 to be declared significantly different. The sample means for the treatments are given below.

	A_1	A_2	A_3
$\bar{y}$	38.4	40.3	41.8

Since the means do not differ by more than 3.703 there are no differences between the treatments, using the Bonferroni procedure.

14.8 Exercises

1. Filby, Y., Szara, S. and Saltzman, B. Magnesium Pemoline Effect on Acquisition and Retention of Discriminated Avoidance Behavior, *Psyconomic Scienc*e, 1967, 9, 131-133 examined how drugs improve learning in rats. Three groups were examined; Magnesium Pemoline (MgPe), d-amphetamine and a placebo. The dependent measure was the speed that rats learned to avoid shock. The data is given below.

MgPe	d-amphetamine	Placebo
7	10	11
8	10	55
8	15	55
9	22	59
10	26	80
27	8	80

   a. Perform the ANOVA and state your conclusions clearly.

   b. Compare the two drug groups in a contrast. Compare the average of the two drug groups with the placebo. Report your results in an ANOVA table.

   c. Are the residuals normal?

2. Grice, G. and Saltz, E. The Generalization of an Instrumental Response to Stimuli Varying in the Size Dimension, *J. Exp.Psychology,* 1950, 40, 705. The psychologists studied the relationship between test area and the number of responses. Four areas were considered with 15 people within each group. The dependent variable was the number of responses to 25 test trials.

Area			
20	32	50	79
9	22	8	12
19	13	11	4
10	17	2	1
21	20	3	8
10	8	4	14
18	22	6	14
11	12	10	5
18	8	13	8
23	14	15	4
10	9	10	11
10	8	4	3
10	14	10	5
9	13	4	5
10	16	8	4
8	4	1	0

a. Perform the ANOVA and state your conclusions clearly.
b. Compare the 20 vs. 32 group in a contrast. Compare the 50 vs. 79 groups in a contrast. Compare the average of the 20 and 32 vs. the average of the 50 and 79 in a contrast. Write out the contrast matrix for the above. Report your results in an ANOVA table.

c. Are the residuals normal?

d. Is there a linear, $E(y|x) = \beta_1 + \beta_2 x$ relationship between response and area? Display your results for the design in an ANOVA table.

3. Lowe, R. (1935), in Cochran (1958), gives the amount of fat absorbed by doughnuts when cooking. The type of fat had 4 levels (1, 2, 3, 4). The data is given below.

Grams of Fat Absorbed per Batch (minus 100 grams)

Fat	1	2	3	4
	64	78	75	55
	72	91	93	66
	68	97	78	49
	77	82	71	64
	56	85	63	70
	95	77	76	68

a. Perform the ANOVA analysis and state your conclusions clearly.

b. Compare F_1 with F_2, F_1 and F_2 with F_3, and F_1 and F_2 and F_3 with F_4. Are the contrasts orthogonal? Report your results in an ANOVA table and state your conclusions.

c. Are the residuals reasonable?

4. Jackson, L.M. and Gorassini, D.R. (1989). *Journal of General Psychology, 116(4),* 333-343 examined 4 treatments; the result of crossing 2 factors, Suggestibility and Condition each at two levels. Participants were classified as high or low with respect to hypnotic suggestibility. Participants also took a creativity test either under hypnosis or in a waking state/condition. A number of dependent measures were examined and are given below. Column A gives the four treatments where:

1 = low suggestibility	with task motivation
2 = high suggestibility	with task motivation
3 = low suggestibility	with hypnosis
4 = high suggestibility	with hypnosis

The data for the dependent variables is given below for Figural Fluency (H), Verbal Fluency (E), Self Report Creativity (M) and Column Experience (O).
For each variable (E, H, M, O)

a. Perform a one-way analysis of variance on the four treatments.

b. Compare via contrasts
   i. Treatments 1 and 2 vs. 3 and 4
   ii. Treatments 1 and 3 vs. 2 and 4
   iii. Treatments 1 and 4 vs. 2 and 3

c. Interpret each of the above contrasts.

d. Comment on the residuals

e. State your conclusions clearly.

ID	A	E	H	M	O	ID	A	E	H	M	O
1	1	27	7	7	9	31	3	26	3	9	21
2	1	10	2	6	10	32	3	30	5	7	16
3	1	18	4	12	17	33	3	32	6	17	20
4	1	38	4	13	24	34	3	25	6	5	10
5	1	27	3	15	14	35	3	28	4	10	13
6	1	31	3	8	17	36	3	37	1	12	14
7	1	28	4	10	16	37	3	39	5	5	19
8	1	19	3	12	22	38	3	30	7	4	12
9	1	43	2	15	19	39	3	12	5	8	8
10	1	32	3	12	16	40	3	30	9	10	15
11	1	70	5	7	14	41	3	34	5	5	21
12	1	66	5	14	20	42	3	29	4	8	12
13	1	35	4	15	20	43	3	29	4	8	12
14	1	48	6	13	16	44	3	19	3	6	11
15	1	47	5	13	21	45	3	37	5	10	14
16	2	25	1	10	7	46	4	32	6	10	19
17	2	34	6	10	18	47	4	30	6	6	18
18	2	37	3	6	20	48	4	53	7	6	18
19	2	37	4	7	18	49	4	27	4	15	20
20	2	33	2	6	7	50	4	21	10	5	10
21	2	32	3	9	10	51	4	30	8	16	22
22	2	23	2	6	17	52	4	31	10	5	21
23	2	32	5	3	11	53	4	25	4	10	7
24	2	38	8	11	23	54	4	25	4	14	21
25	2	24	5	10	9	55	4	41	4	16	18
26	2	66	7	14	14	56	4	39	4	15	21
27	2	42	4	10	17	57	4	38	10	13	22
28	2	39	10	10	18	58	4	38	4	6	18
29	2	27	6	6	17	59	4	27	4	10	17
30	2	73	7	14	23	60	4	28	10	12	20

14.9 ORTHOGONAL POLYNOMIALS

We have learned how to fit models of the type

$$\boxed{E(y|x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \ldots + \beta_k x^{k-1}}$$

which represent polynomials in chapter 8 where x and y were quantitative variables. In the simplest situation of the single factor ANOVA; if the X values are quantitative and equally spaced the polynomial curve fitting can be easily accomplished with the use of contrasts.

The polynomial model above can be rewritten as

$$y' = A_0\xi_0 + A_1\xi_1 + \ldots + A_k\xi_k$$

where each ξ_j is a polynomial of degree j, and all polynomials such as ξ_r and ξ_s are orthogonal. Fisher and Yates (1953) have shown that the advantage of writing the model this way is that polynomials of higher degree can be added which are orthogonal (independent) of the ones already considered. The highest order polynomial is tested first and the psychologist stops when the corresponding contrast is significant. This procedure is very efficient and useful in finding the degree of the polynomial model needed for the psychological system provided X is quantitative and equally spaced.

Fisher and Yates (1953) have provided tables of contrasts for polynomial models. A brief table of coefficients is given in table 14.1. Psychologists sometimes refer to this procedure of curve fitting as a *trend analysis*.

The natural ordering of highest order contrast to lowest order is tested sequentially and we stop when we obtain a significant result. Note that this sequence of nested hypothesis was discussed in chapter 11 of the linear model. For k treatments in the one factor ANOVA we have the following hypothesis.

H_k: $\xi_k = 0$
H_{k-1}: $\xi_{k-1} = 0$ assuming $\xi_a = 0$
.
.
.
etc.

The ANOVA table is given below

Source	DF
ξ_k (contrast 1)	1
ξ_{k-1} given $\xi_k = 0$ (contrast 2)	1
.	.
.	.
.	.
Error	N-k
Total	N-1

The calculations of Sums of Squares, degrees of freedom etc. are identical to the previous section describing the 1-Factor design.

Table 14.1 Table of Coefficients of orthogonal polynomials

a	Polynomial	Coefficients				
3	Linear	-1	0	1		
	Quadratic	1	-2	1		
4	Linear	-3	-1	1	3	
	Quadratic	1	-1	-1	1	
	Cubic	-1	3	-3	1	
5	Linear	-2	-1	0	1	2
	Quadratic	2	-1	-2	-1	2
	Cubic	-1	2	0	-2	1
	Quartic	1	-4	6	-4	1

Note that a refers to the number of treatments.

14.10 Example

Grant, D. and Schiller, J., Generalization of the Conditioned Galvanic Skin Response to Visual Stimuli, *J. Exp. Psychology,* 1953, 46, 309-313 examined the conditioning of the Galvanic Sking Response (GSR) to visual stimuli. The unconditioned stimulus was electric shock; the conditioned stimulus was a rectangle of white light. Seven stimuli were employed at heights of 9, 10, 11, 12, 13, 14 and 15 inches. 14 people were conditioned in each of the 7 stimuli groups for a total of 98 people. The people were then exinguished with a different stimulus and the magnitudes of the GSR in log-conductance units was recorded. The data are given below.

Individual GSR Magnitudes in Log-Conductance Units
For the Various Lengths of the Test Light

	9	**10**	**11**	**12**	**13**	**14**	**15**
	1.57	8.00	3.83	1.02	11.24	3.48	4.61
	.00	4.58	7.23	2.43	4.63	3.63	2.04
	1.20	.00	.48	.00	3.20	8.79	3.90
	.95	2.39	7.25	5.69	5.24	2.67	5.63
	2.24	2.31	6.97	2.84	2.67	2.22	3.72
	2.60	.54	4.22	1.58	1.61	.00	3.62
	2.50	2.31	1.20	3.97	3.97	.00	.00
	3.29	3.62	2.89	6.78	5.87	3.63	.95
	1.15	1.61	1.80	3.72	3.72	.00	6.12
	.00	1.89	6.11	7.84	3.08	3.28	9.01
	.75	.37	4.65	1.22	1.54	1.29	1.61
	.95	.00	1.74	7.15	1.13	3.72	3.72
	1.03	1.53	1.60	4.49	2.89	2.69	6.78
	3.11	7.91	10.35	5.26	3.31	3.63	.15
Totals	21.34	37.06	59.87	54.09	54.10	39.03	51.86
Means	1.52	2.65	4.28	3.71	3.86	2.86	3.70

There are seven levels to the HEIGHT factor therefore 7-1 = 6 degrees of freedom available for contrasts. The following table of orthogonal polynomials is obtained from Appendix A.

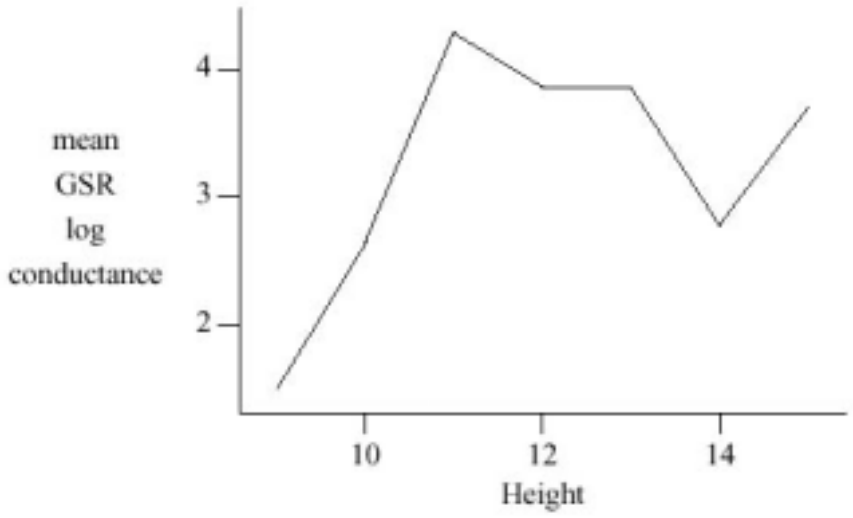
Orthogonal Coefficients

Height	9	10	11	12	13	14	15	$\sum \zeta_i^2$
Linear ξ_1	-3	-2	-1	0	1	2	3	28
Quadratic ξ_2	5	0	-3	-4	-3	0	5	84
Cubic ξ_3	-1	1	1	0	-1	-1	1	6
Quartic ξ_4	3	-7	1	6	1	-7	3	154
Quintic ξ_5	-1	4	-5	0	5	-4	1	84
Sextic ξ_6	1	-6	15	-20	15	-6	1	924

Performing the calculations described in the previous 1-Factor design section gives the following ANOVA table.

Source	DF	SS	MS	F	P
Between heights	6	74.80	12.47	2.14	.056
Linear	1	21.47	21.47	3.69	.06
Quadratic	1	28.75	28.75	4.94	.03
Cubic	1	13.22	13.22	2.27	.14
Quartic	1	5.20	5.20	.89	.35
Quintic	1	.09	.09	.02	.90
Sextic	1	6.08	6.08	1.04	.31
Error	91	529.41			
Total	97				

One sequentially tests from the highest order polynomial (in this example, sextic) and continues to test lower orders of polynomial if the higher is not significant. The data in this example clearly support a quadratic trend to the data. $F(1,91) = 4.94$, $p < .05$. A plot of the means is given below



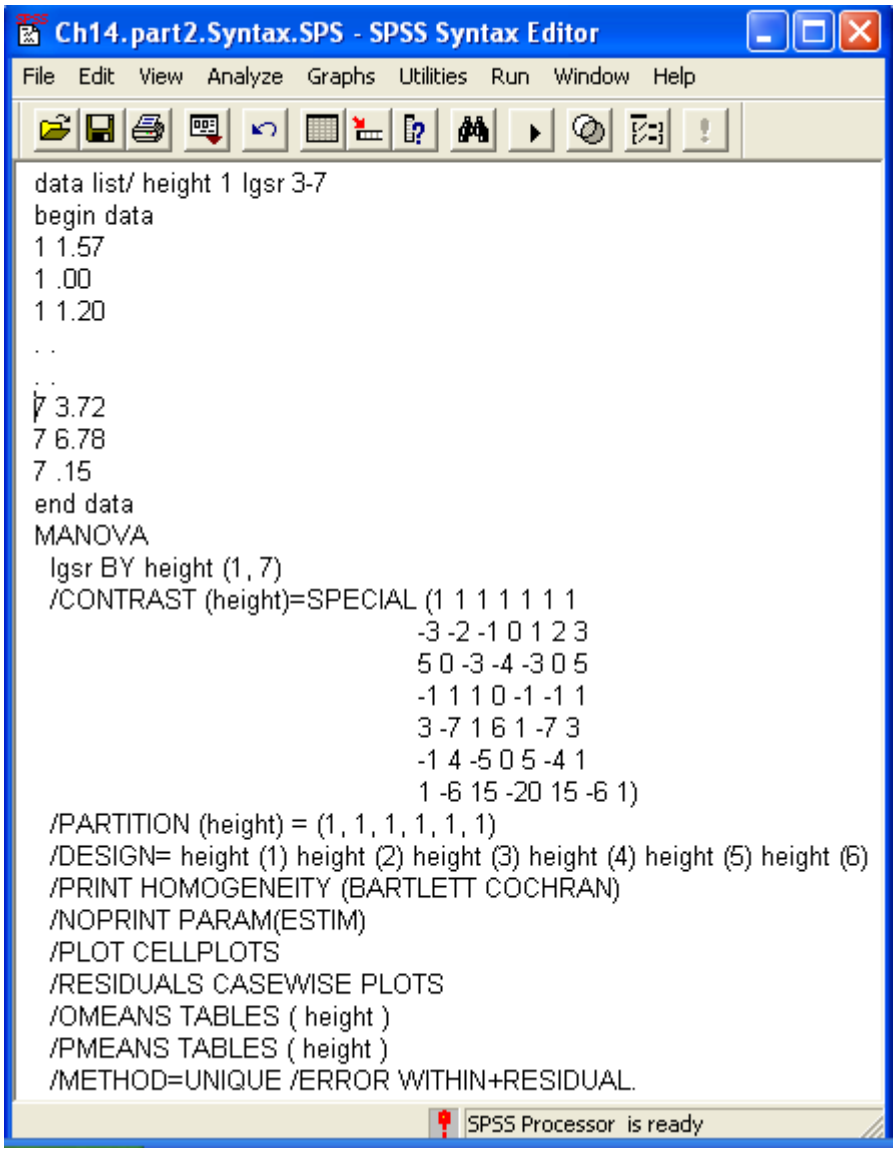
The equation of the quadratic can now be obtained using formulas provided by Fisher and Yates or using the polynomial regression techniques used previously.

14.11 COMPUTER IMPLEMENTATION

The following commands will implement the one way ANOVA analysis described above using the MANOVA procedure in SPSS. Note that the only changes to the MANOVA procedure learned previously in section 14.5 are the contrasts necessary for a trend analysis.

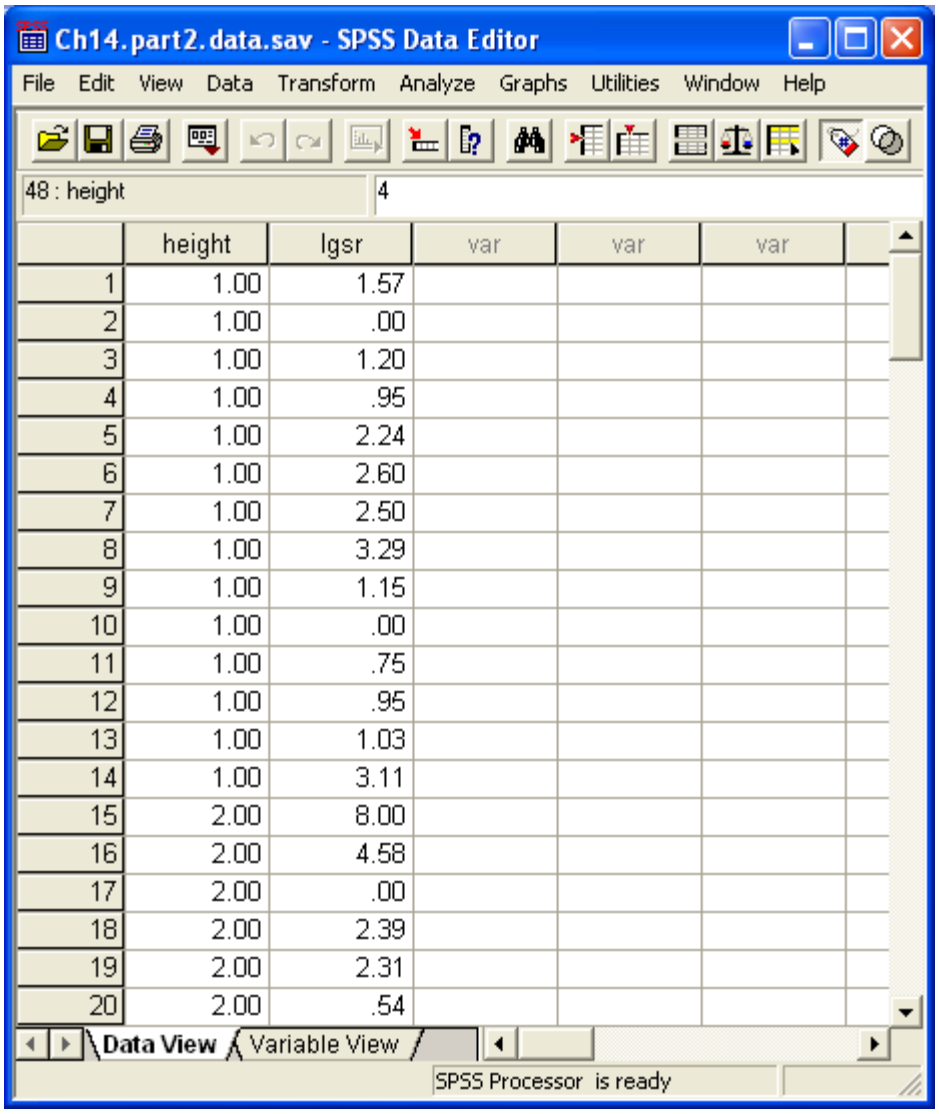
Like the examples outlined in 14.5, you have two choices regarding how to enter in your data when running a syntax program:

1. If you want to enter the data directly into the syntax window, please follow these steps:
    a) Click **File** on the menu bar, then **New**, followed by **Syntax**. This series of clicks will open a SPSS Syntax Editor window. Type the following commands into the Syntax Editor. Be sure to type the Syntax exactly as you see it in the picture provided below (i.e., use the appropriate case, spaces between words, etc.).

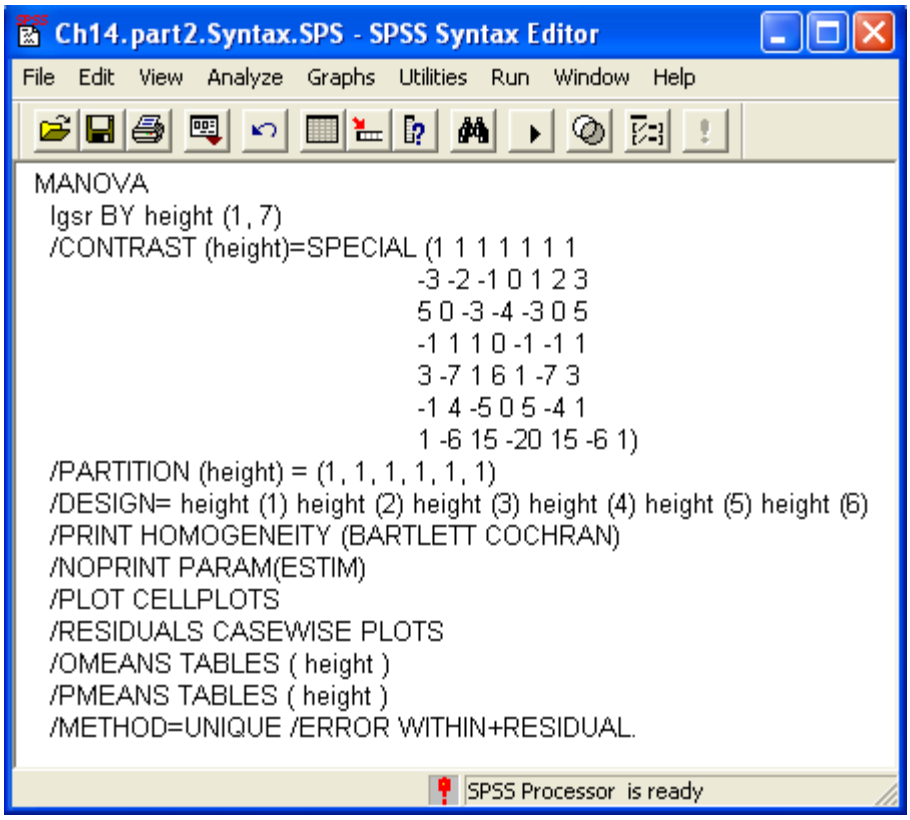


b) When you have completed typing the above commands into the Syntax window, click on **Run**, and then **All**. Your output will then be available for you to view.

2. If you prefer to enter you data in separately from your syntax, please follow these steps:
   a) Open up SPSS and enter your data into the **Data Editor** (the main data page). Your data and a picture of the data editor have been provided below:



b) Now you can enter your commands into the Syntax Window. Click **File** on the menu bar, then **New**, followed by **Syntax**. This series of clicks will open a SPSS Syntax Editor window. Type the following commands into the Syntax Editor. Be sure to type the Syntax exactly as you see it in the picture provided below (i.e., use the appropriate case, spaces between words, etc.).



c) When you have completed typing the above commands into the Syntax window, click on **Run**, and then **All**. Your output will then be available for you to view.

The means are output for each height and given below.

```
Combined Observed Means for HEIGHT
 Variable .. LGSR
        HEIGHT
             1        WGT.      1.52429
                     UNWGT.     1.52429
             2        WGT.      2.64714
                     UNWGT.     2.64714
             3        WGT.      4.27643
                     UNWGT.     4.27643
             4        WGT.      3.85643
                     UNWGT.     3.85643
             5        WGT.      3.86429
                     UNWGT.     3.86429
             6        WGT.      2.78786
                     UNWGT.     2.78786
             7        WGT.      3.70429
                     UNWGT.     3.70429
```

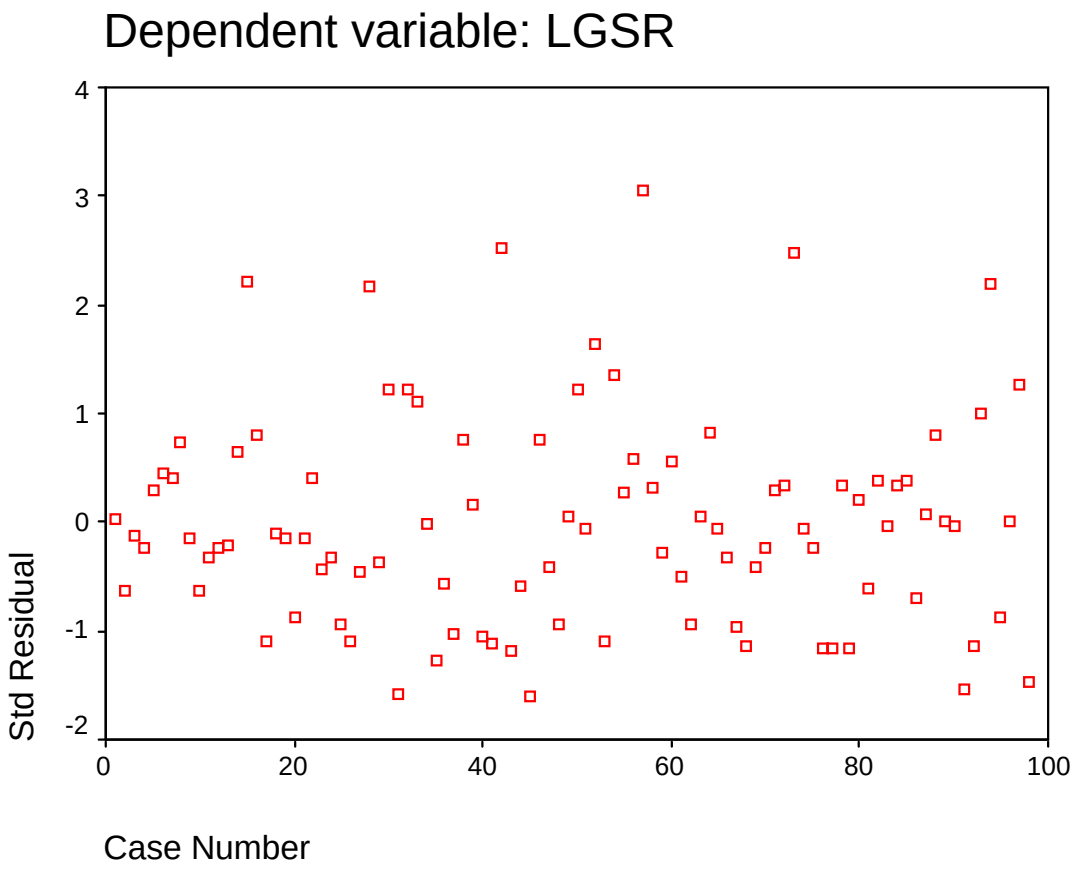
The fine ANOVA table is given below. Begin testing with the highest order polynomial (HEIGHT(6) F=.80, p=.372) and continue until a significant effect is detected. Stop testing at the (HEIGHT(2)) term since it is the highest order significant term (at ⟨=.05), F(1,91) = 5.37, p = .023. The trend is therefore quadratic.

```
Tests of Significance for LGSR using UNIQUE sums of squares
 Source of Variation          SS      DF        MS         F  Sig of F

 WITHIN+RESIDUAL          530.61      91      5.83
 HEIGHT(1)                 20.54       1     20.54      3.52      .064
 HEIGHT(2)                 31.30       1     31.30      5.37      .023
 HEIGHT(3)                 14.02       1     14.02      2.40      .124
 HEIGHT(4)                  7.23       1      7.23      1.24      .268
 HEIGHT(5)                   .03       1       .03       .01      .940
 HEIGHT(6)                  4.69       1      4.69       .80      .372

 (Model)                   77.83       6     12.97      2.22      .048
 (Total)                  608.44      97      6.27

 R-Squared =           .128
 Adjusted R-Squared =  .070
```

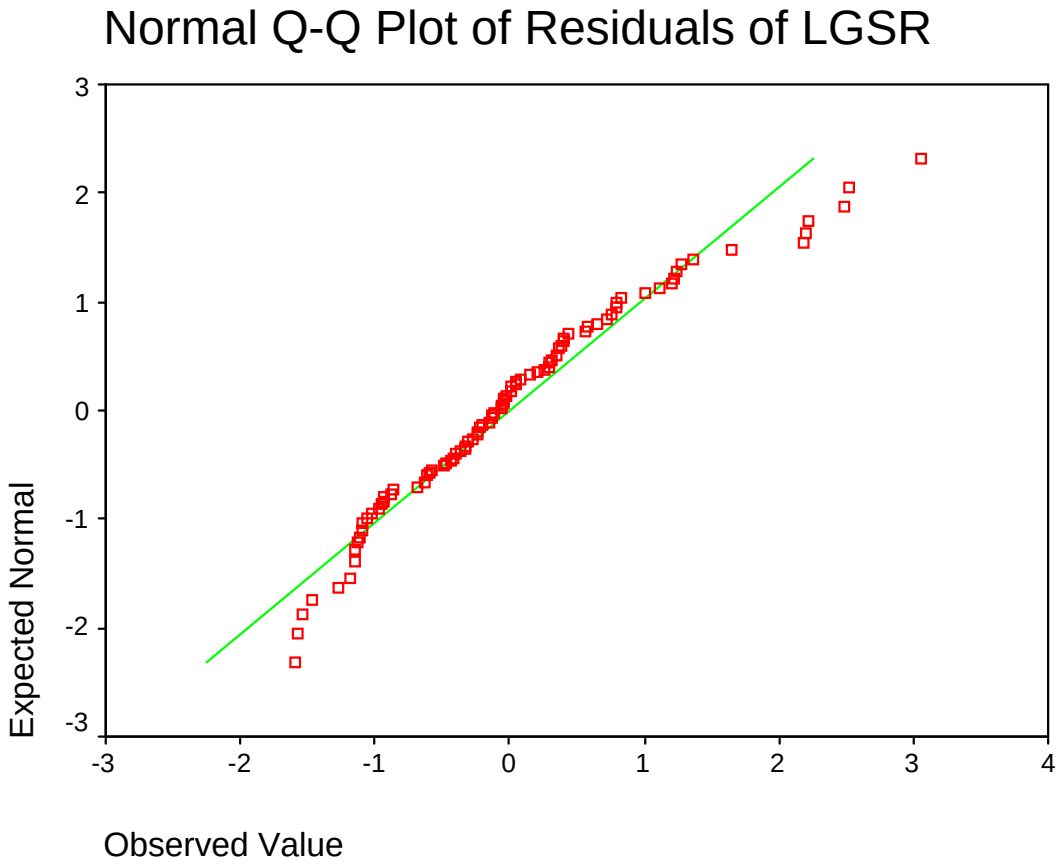
The crude ANOVA table is given below. Note that the HEIGHT variable is significant, F(6,91) = 2.22, p = .048.

```
Tests of Significance for LGSR using UNIQUE sums of squares
 Source of Variation          SS      DF        MS         F  Sig of F

 WITHIN+RESIDUAL          530.61      91      5.83
 HEIGHT                    77.83       6     12.97      2.22      .048

 (Model)                   77.83       6     12.97      2.22      .048
 (Total)                  608.44      97      6.27

 R-Squared =           .128
 Adjusted R-Squared =  .070
```

The plot of case number vs. standardized e_i looks reasonable displaying a band pattern.



The normal probability plot is reasonable, since it approximates a line, however there is a slight bend in it.



SPSS has built into its software some common contrasts which we will use in future chapters.
Click here for a summary of the types of contrasts associated with each SPSS command.

14.12 SAS COMPUTER IMPLEMENTATION

An SAS program to implement the above trend analysis is given below.

```
DATA TREND;
INPUT HEIGHT LGSR;
CARDS;
1 1.57
1 .00
1 1.20
..
..
..
7 .15
PROC GLM;
CLASS HEIGHT;
MODEL LGSR = HEIGHT;
MEANS HEIGHT/;
```

```
CONTRAST 'LINEAR' HEIGHT -3 -2 -1 0 1 2 3;
CONTRAST 'QUADRATIC' HEIGHT 5 0 -3 4 -3 0 5;
CONTRAST 'CUBIC' HEIGHT -1 1 1 0 -1 -1 1;
CONTRAST 'QUARTIC' HEIGHT 3 -7 1 6 1 -7 3;
CONTRAST 'QUINTIC' HEIGHT -1 4 -5 0 5 -4 1;
CONTRAST 'SEXTIC' HEIGHT 1 -6 15 -20 15 -6 1;
OUTPUT OUT=RESIDS P=YHAT R=RESID;
PROC PLOT;
PLOT RESID*(YHAT,CRIME);
PLOC UNIVARIATE PLOT NORMAL;
VAR RESID;
```

14.13 Exercises

1. Kas, K. and Dember, W., Effects of Size of Ring on Backward Masking of a Disk by a Ring, *Psychonomic Science,* 1973, 2, 15-17, studied backward masking. If a black disk appears for a short time, followed by a ring where the inner edge corresponds to the outside of the disk, you may never perceive the disk, only the ring. The authors investigated ring thickness and how it affected perception. The data for 20 people is based on the Kas study and are listed below.

	Ring Thickness (mm)			
0	.25	.5	1.0	1.5
4.69	9.17	26.21	27.14	27.73
13.02	16.35	21.56	19.52	19.93
4.05	13.17	16.98	27.25	27.02
5.73	15.25	18.01	20.92	29.85

   a. Perform the ANOVA analysis. Comment on the distribution of the residuals.

   b. Use orthogonal polynomials to determine the type of trend exhibited by the data. Omit the .25 results for this analysis. Report your results in an ANOVA table.

2. Sheffield, V., Extinction as a Function of Partial Reinforcement and Distribution of Practice, *J. Exp. Psychol.* 1949, 39, 511-526, examined how learning was affected by percentage of reward in rats. Four treatments, percentage of trials rewarded (25%, 50%, 75% and 100%) were considered. The dependent variable was the number of extinction trials required under each. The data for 40 rats, 10 per treatment is based on Sheffield's study.

Reward			
25%	50%	75%	100%
10	12	14	9
14	22	18	21
18	20	21	18
20	16	10	17
10	9	13	10
9	15	9	15
15	18	14	11
13	17	14	16
8	13	9	14
9	14	12	7

a. Perform the appropriate ANOVA analysis.

b. Determine the type of trend displayed by the data. Report your results in an ANOVA table.