# 15. THE DIRECT PRODUCT AND INTERACTIONS

## The Direct Product

We define the Direct Product of two vectors as follows:

**Def:** If x is a vector of m elements and y is a vector of n elements then the direct product of x and y denoted by

$$\mathbf{x} \otimes \mathbf{y} = (x_1y_1, x_2y_2, \ldots, x_my_n)$$

which has *mn* elements
i.e. $\mathbf{x}$ =(12) m=2
$\mathbf{y}$ = (1 2 3) n = 3
$\mathbf{x} \otimes \mathbf{y}$= (1.1 1.2 1.3 2.1 2.2 2.3) = (12 3 2 4 6)

## 15.1 INTERACTION CONTRASTS

Suppose we have two factors $A{:}A_1$, $A_2$ and $A_2$ and $B{:}B_1$, $B_2$ ie. call this a 2x2 factorial. Then the contrast matrix for factor A is given by

$$C_A = (C_{11}\ C_{21}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

and the contrast matrix for factor B is

$$C_B = (C_{12}\ C_{22}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

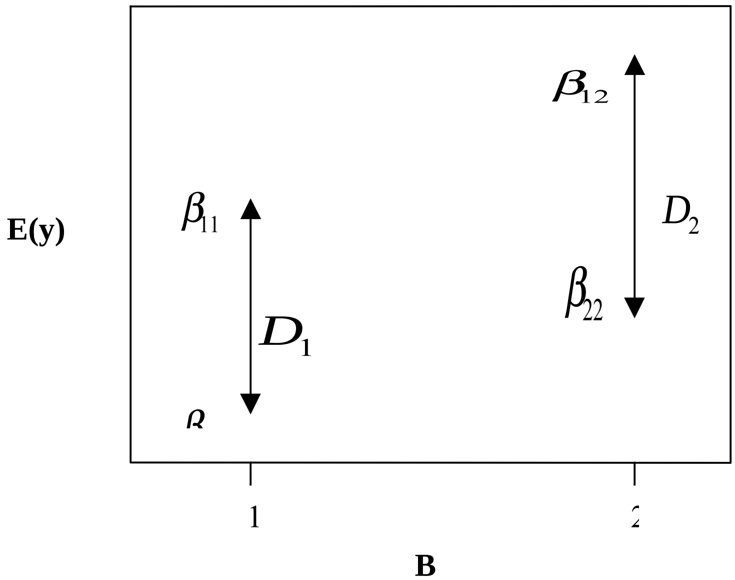
Now the model for a single response is

$$E\,[y] = \beta_{11} x_{11} + \beta_{21} x_{21} + \beta_{12} x_{12} + \beta_{22} x_{22}$$

where $x_{ij}$ =1 if the response is from treatment $A_iB_j$, 0 otherwise

$\beta_{ij}$ is the mean of the frequency distribution obtained from applying $A_iB_j$

**Figure 15.1**




Before checking whether or not effects due to A or B exist we must first check to see whether or not these factors interact. (Refer to Chapter 5 for a preliminary discussion of the nature of interaction and the role of differences.)

In order to do this we must examine the difference in the difference of the mean responses to factor A between the two levels of factor B; namely.

$$(\beta_{11} - \beta_{21}) - (\beta_{12} - \beta_{22}) = D_1 - D_2$$

which also equals

$$(\beta_{11} - \beta_{12}) - (\beta_{21} - \beta_{22}) = D_2 - D_3$$

The difference in the difference of the mean response to factor B between the two levels of factor A.

We note that this is equivalent to testing whether or not
$(\beta_{11} - \beta_{21})/2 - (\beta_{12} - \beta_{22})/2$
i.e. $(C_{21} \otimes C_{22})\,\beta$ is equal to zero (0)

$$(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) \otimes (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}$$

We refer to $C_{21} \otimes C_{22}$ as the interaction contrast.

If no interaction exists between A and B then we can examine whether an effect exists due to A by testing whether

$$\frac{\beta_{11}-\beta_{21}}{2}+\frac{\beta_{12}-\beta_{22}}{2}$$

$= (C_{21} \otimes C_{12})\, \beta = 0$

Further we test whether or not an effect due to B exists by testing whether

$$\frac{\beta_{11}-\beta_{12}}{2}+\frac{\beta_{21}-\beta_{22}}{2}$$

$$= \frac{\beta_{11}-\beta_{21}}{2}-\frac{\beta_{12}-\beta_{22}}{2}$$

$= (C_{11} \otimes C_{22})\beta = 0$

We see that the relevant parameters are obtained by forming direct products of the rows of *CA* and CB. In effect we have found the contrast.272 Linear Models and Experimental Design matrix **C** for the 2 factor design where:

$C = (C_{11} \otimes C_{12} \quad C_{11} \otimes C_{22} \quad C_{21} \otimes C_{12} \quad C_{21} \otimes C_{22})$

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

or

Divisor

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix}$$

The parameters for the 2x2 factorial model and corresponding coefficients, divisors and interpretation is given below.

| $\beta_{11}$ | $\beta_{21}$ | $\beta_{12}$ | $\beta_{22}$ | Divisor | Interpretation |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | Mean |
| 1 | -1 | 1 | -1 | 2 | Effect of A |
| 1 | 1 | -1 | -1 | 2 | Effect of B |
| 1 | -1 | -1 | 1 | 2 | Interaction AxB |

The table below identifies the different treatments as indexed by subscripts. Notice how the contrast coefficients and paramteres are directly linked to their interpretations.

| | | B: 1 | B: 2 |
|---|---|---|---|
| A | 1 | $\beta_{11}$ | $\beta_{12}$ |
| | 2 | $\beta_{21}$ | $\beta_{22}$ |

As we have discussed previously, the Effect of A is an average of treatments that received $A_1$ ($\beta_{11}$ and $\beta_{12}$) vs the average of treatments that received $A_2$ ($\beta_{21}$ and $\beta_{22}$).

## 15.2 Example One

Consider the case of two factors A and B. Call this an a x b factorial.

(a x b factorial) A: $A_1, \ldots A_a$.
B: $B_1, \ldots B_b$

then if $C_1 = (C_{11}\ C_{21}\ \ldots\ C_{a1})$ is an orthogonal contrast matrix for A
and $C_2 = (C_{12}\ C_{22}\ \ldots\ C_{b1})$ is an orthogonal matrix for B

$$C = C_1 \otimes C_2$$
$$= (C_{11} \otimes C_{12} \quad C_{11} \otimes C_{22} \quad \ldots \quad C_{a1} \otimes C_{b2})$$

is an orthogonal contrast matrix for the 2 factor design where

| | |
|---|---|
| $C_{i1} \otimes Cj_2$ | for i, j $\neq 1$ are interaction contrasts |
| $C_{j1} \otimes C_{12}$ | i $\neq 1$ are treatment contrasts for A |
| $C_{11} \otimes C_{j2}$ | j $\neq 1$ are treatment contrasts for B |

NOTE in an (a x b) factorial design
there are (a-1)(b-1) interaction contrasts
(a-1) treatment contrasts due to A
(b-1) treatment contrasts due to B

Chapter 16 will explain the reasons for these numbers in detail.

**Example One (continued)**

A clinical psychologist would like to examine at the satisfaction of her clients with three drug addiction programs. Three types of program are evaluated, two Behavioural, one Gestalt. The researcher wishes to compare the two behavioural therapies, and then if there is no difference; compare the average of the two to the Gestalt. The researcher also wishes to examine the effect on satisfaction of the use of volunteers. She examined programs that utilized volunteers and those that did not.
Factor A - use of volunteers or not.
Factor B - Type of Therapy, Gestalt, Behavioural 1, Behavioural 2

$C_A$ is the contrast matrix for Factor A
$C_B$ is the contrast matrix for Factor B
$C = C_A \otimes C_B$ is the overall design matrix.

The matrices $C_A$, $C_B$ can be written as follows
$C_A = ( C_{11}, C_{21})$

$C_B = ( C_{12}, C_{22}, C_{32})$

The actual contrasts are given below for each matrix.

$$C_A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{matrix} C_{11} \\ \cdot \\ C_{12} \end{matrix} \qquad C_B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{matrix} C_{12} \\ \cdot \\ C_{22} \\ \cdot \\ C_{32} \end{matrix}$$

The overall design matrix $\mathbf{C = C_A \otimes C_B}$ is given below.

$$\begin{matrix} C_{11} \otimes C_{12} \\ C_{11} \otimes C_{12} \\ C_{11} \otimes C_{12} \\ C_{11} \otimes C_{12} \\ C_{11} \otimes C_{12} \\ C_{11} \otimes C_{12} \end{matrix} \begin{vmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{-2}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{2} \end{vmatrix}$$

The table below identifies the different parameters in the model and corresponding contrast coefficients.

| Interpretation | $\beta_{11}$ | $\beta_{12}$ | $\beta_{13}$ | $\beta_{21}$ | $\beta_{22}$ | $\beta_{23}$ | Divisor |
|---|---|---|---|---|---|---|---|
| Mean | 1 | 1 | 1 | 1 | 1 | 1 | $\sqrt{6}$ |
| Effect $B2$ | 2 | -1 | -1 | 2 | -1 | -1 | $\sqrt{12}$ |
| Effect $B1$ | 0 | 1 | -1 | 0 | 1 | -1 | 2 |
| Effect A | 1 | 1 | 1 | -1 | -1 | -1 | $\sqrt{6}$ |
| Effect $AB2$ | 2 | -1 | -1 | -2 | 1 | 1 | $\sqrt{12}$ |
| Effect $AB1$ | 0 | 1 | -1 | 0 | -1 | 1 | 2 |

The $B_2$ contrast is a comparison of the average of the two Behavioural groups vs the Gestalt group. The $B_1$ contrast is a comparison of the mean of one Behavioural group vs the other. The A contrast compares the mean of the Volunteer vs the no volunteer group. The $AB_2$ interaction contrast compares the difference of the two Behavioural groups vs the Gestalt group for volunteers with the same difference for non volunteers. The $AB_1$ interaction contrast compares the difference of the two Behavioural groups for volunteers with the same difference for non volunteers.

### 15.3 Example Two

Consider the case of three factors, A, B and C, an a x b x c factorial.

(a x b x c) factorial
A: $A_1, \ldots, A_a$
B: $B_1, \ldots, B_b$
C: $C_1, \ldots, C_c$

In this case one can have 2nd order interactions as well (ie. differences of differences, see Chapter 5).
Note that just as we checked for first order interactions by examining contrasts of contrasts we check for second order interactions by examining contrasts of contrasts of contrasts.

Let $C_1 = (C_{11}, \ldots, C_{a1})$
$C_2 = (C_{12}, \ldots, C_{b1})$
$C_3 = (C_{13}, \ldots, C_{c1})$
be orthogonal contrasts matrices for A, B and C respectively
then $\mathbf{C} = C_1 \otimes C_{11} \otimes C_{a1}$
$= (C_{11} \otimes C_{12} \otimes C_{13} \quad C_{21} \otimes C_{12} \otimes C_{13} \ \ldots \ C_{a1} \otimes C_{b2} \otimes C_{c3})$
is an orthogonal contrast matrix
- The vectors
$C_{i1} \otimes C_{j2} \otimes C_{k3} \qquad i \neq 1, j \neq 1, k \neq 1$
are 2nd order interaction contrasts.
There are (a-1)(b-1)(c-1) of these.
- The vectors
$C_{i1} \otimes C_{j2} \otimes C_{k3} \qquad i \neq 1, j \neq 1$ are
first order interaction contrasts of factor A over factor B.
There are (a-1)(b-1) 1st order interaction contrasts of A over B

- The vectors
$C_{i1} \otimes C_{j2} \otimes C_{k3}$ $i \neq 1, k \neq 1$ are
first order interaction contrasts of factor B over factor C.
There are (b-1)(c-1) 1st order interaction contrasts.
- The vectors
$C_{i1} \otimes C_{12} \otimes C_{k3}$ $i \neq 1, k \neq 1$ are
first order interaction contrasts of factor A over factor C.
There are (a-1)(c-1) 1st order interaction contrasts.
- The vectors
$C_{i1} \otimes C_{12} \otimes C_{k3}$ $i \neq 1$ are the treatment contrasts for factor A.
There are (a-1) treatment contrasts for factor A.
- The vectors
$C_{i1} \otimes C_{j2} \otimes C_{k3}$ $j \neq 1$ are the treatment contrasts for factor B. There are (b-1) treatment contrasts for factor B.
- The vectors
$C_{i1} \otimes C_{12} \otimes C_{k3}$ $k \neq 1$ are the treatment contrasts for factor C. There are (c-1) treatment contrasts for factor C.

**Example Three-click here for another example**

**15.4 Exercises**

1. Consider the following two factor experiment based on Gebotys and Roberts, *Canadian Journal of Behovioral Science, 1987,* 19, 479-488. CRIME is a factor with 3 levels (break and enter, sexual assault and manslaughter) and AGE at 4 levels (20, 30, 40, 50 years old). Compare the following crimes using contrasts, $C_1$ *vs* $C_2$ ; $C_1$ and $C_2$ *vs* $C_3$ . Compare the following ages, $A_1$ *vs* $A_2$; $A_1$ and $A_2$ *vs* $A_3$ ; $A_1$ and $A_2$ and $A_3$ *vs* $A_4$ . Give the separate contrast matrices for AGE and CRIME. Give the overall design matrix and interpret each contrast.

2. Suppose a developmental psychologist wished to conduct the following 3 factor study. Factor one was GENDER at 2 levels (M vs. F), factor two was LOCATION at 2 levels (city vs. country) and factor three was PROGRAM at 3 levels (easy, medium, hard). Give the contrast matrix for each of GENDER, LOCATION and PROGRAM (where the contrast she is interested in is easy vs. medium, and easy and medium vs. hard). Give the overall design matrix as well and interpret each contrast.