18. 3^k FACTORIAL DESIGNS

These designs are often used when we have k quantitative variables and we want to fit a polynomial in all the variables. To simplify matters we design the experiment so that each factor takes 3 equispaced values. The orthogonal polynomials for a single variable, taking values 1, 2, 3 are given by

Linear	-1	0	1
Quad	1	-2	1

then for example the Design matrix for a 3^2 factorial (suppose n = 1) is given by.

$$\left|\begin{array}{cccccccccc}
\text{Mean} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
B_L & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\
B_Q & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\
A_L & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\
A_Q & 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\
A_L B_L & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
A_L B_Q & 1 & -2 & 1 & 0 & 0 & 0 & -1 & 2 & -1 \\
A_Q B_L & 1 & 0 & -1 & -2 & 0 & 2 & 1 & -2 & 1 \\
A_Q B_Q & 1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1
\end{array}\right|$$

The design matrices for A and B are respectively

$$C_A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

the overall design matrix is then

$$\mathbf{C} = \boldsymbol{C_A} \otimes \boldsymbol{C_B}.$$

		A_L	
	1	0	-1
1	1	0	-2
B_L 0	0	0	0
-1	-1	0	1

18.1 TWO FACTORS (1 qualitative - 1 quantitative)

Suppose that factor A is quantitative with levels X_1 , ... , X_a equispaced, factor B is qualitative with levels B_1 ,..., B_b. For example consider an animal learning experiment where A could be the amount of a given drug and B could be the type of drug.

Example of a 4 x 3 x 2 Factorial Experiment (continued).

The ANOVA table is given below:

ANOVA

Source	DF	SS	MS	F
A	3	310.57	103.52	27.24*
B	2	4.07	2.04	
C	1	56.04	56.04	14.75*
A x B	6	6.73	1.12	
A x C	3	5.76	1.92	
B x C	2	5.06	2.53	
A x B x C	6	6.94	1.16	
Error	96	364.80	3.80	
Total	119	759.97		

where * indicated p < .05.

Suppose the factor A is quantitative and equally spaced. Then we can use orthogonal polynomials to fit cubic, quadratic and linear trends for example.

Comparison	Coefficients	Σa^2
Linear	-3 -1 1 3	20
Quadratic	1 -1 -1 1	4
Cubic	-1 3 -3 1	20

Table of cell means:.

	A_1	A_2	A_3	A_4
B_1	37	61	70	80
B_2	38	58	69	75
B_3	31	53	64	82
Σ	106	172	203	237

Then calculate the following:

$$L_A = (-3)(106) + (-1)(172) + 1(203) + 3(237) = 424$$

$$MS_{LA} = \frac{(424)^2}{30(20)} = 299.63$$

$$MS_{QA} = \frac{(-32)^2}{30(4)} = 8.53$$

$$MS_C = \frac{(38)^2}{30(20)} = 2.41$$

18.2 INTERACTION COMPONENTS

We can calculate interactions of quantitative variables in the same way one calculates interactions of qualitative variables. We briefly outline some of the quantitative interactions below.

ANOVA

Source	Sum of Squares		MS	DF
A	310.57		3	
Linear		299.63	1	
Quadratic		8.53	1	
Cubic		2.41	1	
.			.	
.			.	
.			.	
A x B	6.73		6	
Linear x B			2	
Quadratic x B			2	
Cubic x B			2	
.			.	
.			.	
.			.	
A x C	5.76			
Linear x C			1	
Quadratic x C			1	
Cubic x C			1	
.			.	
.			.	
.			.	
Within Treatments			96	
Total			119	

18.3 Example 3^2 Factorial (Both Quantitative)

Raw Data.

		Factor A 1	Factor A 2	Factor A 3
	1	6.3 5.4	8.0 8.0	7.3 7.8
Factor B	2	6.9 6.5	7.5 7.5	8.6 8.3
	3	7.2 7. 0	8.6 8.0	9.0 9.0

Table of Totals and Treatments:

	3	(13) 11.7	(23) 16	(33) 15.2
Factor B	2	(12) 13.4	(22) 15.0	(32) 16.9
	1	(11) 14.2	(21) 17.6	(31) 18.0

1 2

3

Factor A

We obtain the following ANOVA table:

ANOVA

Source	DF	SS	MS	F	P
A	2	11.44	5.72	71.5	.0001
B	2	4.17	2.08	26.0	.0001
A x B	4	1.39	.35	4.4	<.05
Error	9	.70	.08		
Total (corrected)	17				

	Treatment Combinations										
Factor	00	01	02	10	11	12	20	21	22	$\sum a^2$	$\sum T_{a_i}$
A_L	−1	−1	−1	0	0	0	1	1	1	6	10.8
A_Q	1	1	1	−2	−2	−2	1	1	1	18	−7.8
B_L	−1	0	1	−1	0	1	−1	0	1	6	6.9
B_Q	1	−2	1	1	−2	1	1	−2	1	18	2.2
$A_L B_L$	1	0	−1	0	0	0	−1	0	1	4	.3
$A_L B_Q$	−1	2	−1	0	0	0	1	−2	1	12	.3
$A_Q B_L$	−1	0	1	2	0	−2	−1	0	1	12	2.4
$A_Q B_Q$	1	−2	1	−2	4	−2	1	−2	1	36	−8.9
T_{ij}	11.7	13.4	14.2	16.0	15.0	17.6	15.2	16.4	18.0		

The sums of squares for each contrast can then be calculated, i.e.

A_L = -1(11.7) - 1(13.4) - 1(14.2) + 0(16.0) + 0(15.0) + 0(17.6) + 1(15.2) +1(16.9) + 1(18.0) = 10.8

$$SS_{AL} = \frac{(10.8)^2}{6(2)} = 9.7$$

$$A_Q B_Q = -8.9$$

$$SS_{A_Q B_Q} = \frac{(-8.9)^2}{36(2)} = 1.1$$

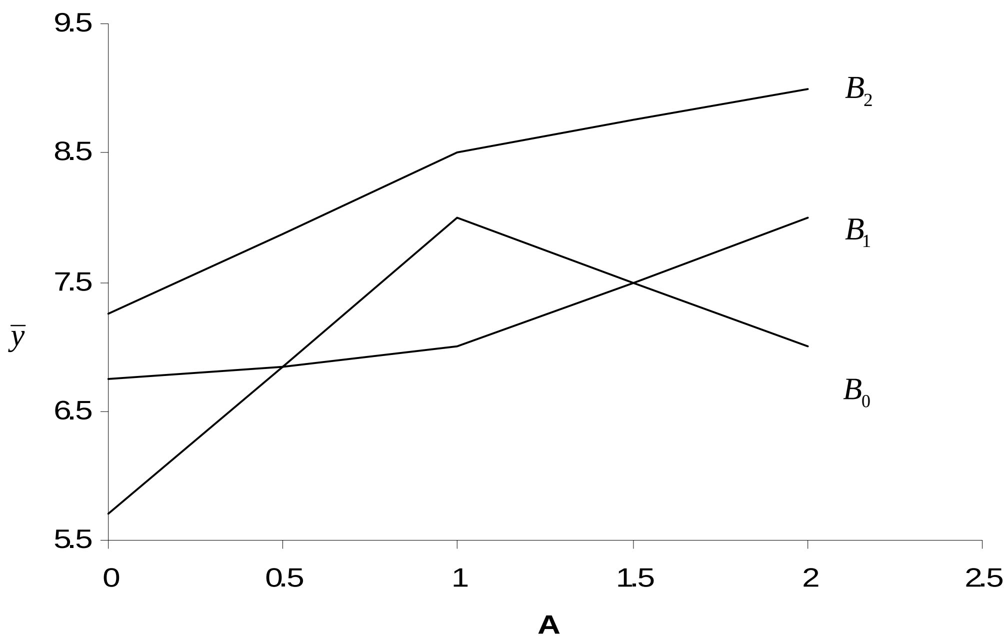
The complete ANOVA table is given below.

ANOVA

Source	DF	SS	F	P
A	2	11.44		
A_L	1	9.7	121.7	<.0001
A_Q	1	1.7	21.2	<.001
B	2	4.17		
B_L	2	4.0	50.3	<.0001
B_Q	1	.1	1.7	
AB	4	1.38		
$A_L B_L$	1	0	.	
$A_L B_Q$	1	0	.	
$A_Q B_L$	1	.2	3.1	
$A_Q B_Q$	1	1.1	14.0	<.01
Error	9	.7		
Totals	17			

Note the $A_Q B_Q$ is significant at p <.01. A graph of the treatment means is given below.

Figure 18.1



The quadratic by quadratic interaction is clearly displayed in the plot. Note the differences are not equal across levels as is the case of an interaction between variables .

18.4 SYNTAX COMPUTER IMPLEMENTATION

```
data list/ A 1 B 3 Y 5-7
begin data
0 0 6.3
0 0 5.4
1 0 8.0
1 0 8.0
2 0 7.3
2 0 7.8
0 1 6.9
0 1 6.5
1 1 7.5
1 1 7.5
2 1 8.6
2 1 8.3
0 2 7.2
0 2 7.0
1 2 8.6
1 2 8.9
2 2 9.0
2 2 9.0
end data
```

If you have already entered the above data into the data editor, next open a syntax file and enter the following:

```
MANOVA
  Y by A (0, 2) B(0, 2)
  /CONTRAST (A)=special(1 1 1
                        -1 0 1
                        1 -2 1)
  /CONTRAST (B)=special(1 1 1
                        -1 0 1
                        1 -2 1)
  /PARTITION (A) = (1, 1)
  /PARTITION (B) = (1, 1)
  /DESIGN=A(1), A(2),
        B(1), B(2),
        A(1) by B(1), A(1) by B(2),
        A(2) by B(1), A(2) by B(2)
  /PRINT HOMEGENEITY (BARTLETT COCHRAN)
  /NOPRINT PARAM(ESTIM)
   /PLOT CELLPLOTS
   /RESIDUALS CASEWISE PLOTS
   /OMEANS TABLES (A B A BY B)
   /PMEANS TABLES ( A B )
  /METHOD=UNIQUE
  /ERROR WITHIN+RESIDUAL.
```

Once you have finished entering the above commands, highlight the entire text and press the small black error located on the tool bar in order to obtain the output. Alternatively you can use the GLM procedure in SPSS for windows that was discussed and demonstrated in chapter 16.

The means are given below.

```
Combined Observed Means for A
 Variable .. Y
             A
             0        WGT.     6.55000
                    UNWGT.     6.55000
             1        WGT.     8.08333
                    UNWGT.     8.08333
             2        WGT.     8.33333
                    UNWGT.     8.33333


Combined Observed Means for B
 Variable .. Y
             B
             0        WGT.     7.13333
                    UNWGT.     7.13333
             1        WGT.     7.55000
                    UNWGT.     7.55000
             2        WGT.     8.28333
                    UNWGT.     8.28333


 Combined Observed Means for A BY B
 Variable .. Y
                         A           0           1           2
             B
             0        WGT.     5.85000     8.00000     7.55000
                    UNWGT.     5.85000     8.00000     7.55000
             1        WGT.     6.70000     7.50000     8.45000
                    UNWGT.     6.70000     7.50000     8.45000
             2        WGT.     7.10000     8.75000     9.00000
                    UNWGT.     7.10000     8.75000     9.00000
```

The fine ANOVA table is given below. The A(2) by B(2) contrast is significant, $F(1,9) = 12.84$, p = .006. A graph of the means would help describe the interaction of these two quadratic effects.

Tests of Significance for Y using UNIQUE sums of squares

Source of Variation	SS	DF	MS	F	Sig of F
WITHIN+RESIDUAL	.72	9	.08		
A(1)	9.54	1	9.54	119.26	.000
A(2)	1.65	1	1.65	20.59	.001
B(1)	3.97	1	3.97	49.59	.000
B(2)	.10	1	.10	1.25	.292
A(1) BY B(1)	.02	1	.02	.25	.629
A(1) BY B(2)	.00	1	.00	.02	.888
A(2) BY B(1)	.24	1	.24	3.00	.117

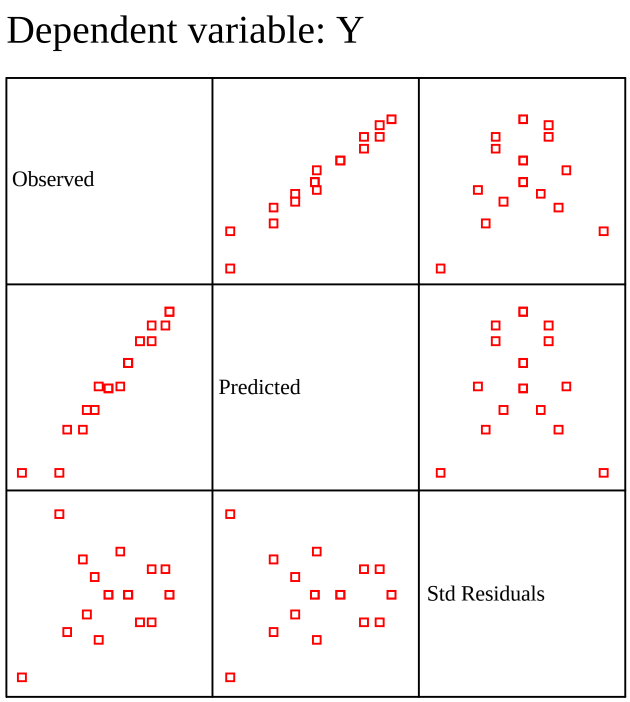
A(2) BY B(2)	1.03	1	1.03	12.84	.006

The coarse ANOVA is given below. All of the terms in the model are significant.

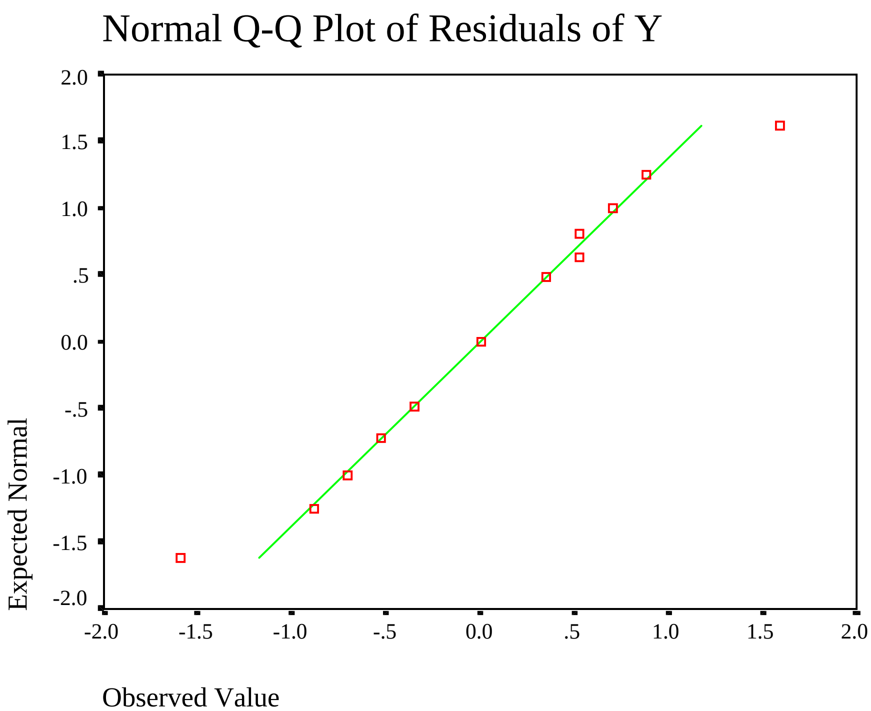
Tests of Significance for Y using UNIQUE sums of squares

Source of Variation	SS	DF	MS	F	Sig of F
WITHIN+RESIDUAL	.72	9	.08		
A	11.19	2	5.59	69.92	.000
B	4.07	2	2.03	25.42	.000
A BY B	1.29	4	.32	4.03	.038

Both residual plots look reasonable. Below is the plot of the Predicted, Observed, and standardized residuals. There is however a bit of a fan shape in the standardized residuals vs predicted plot below.



The normal probability plot, as mentioned above, appears reasonable. This indicates that the residuals are normal.



In summary, the pre planned polynomial contrasts show the reseacher the type of trend in the quantitative data. Just as in ch14 we test the highest order polynomials first. In the case of interactions we test the highest order interactions first. If a significant effect is found the researcher stops the sequence of tests, which has a natural order(high to low order polynomial) and interprets the result using means , graphs, etc..