CHAPTER 6: MATRICES AND LINEAR EQUATIONS

Matrices
A *matrix* is a rectangular array of numbers. The numbers in a matrix are called *element*s. The number of rows and columns are called the *dimensions* of the matrix. Thus, matrix **A**

$$\mathbf{A}_{2x3} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 7 \end{bmatrix}$$

possesses dimensions 2x3, or D(A) = 2x3, because it contains two rows and three columns. Similarly for

$$\mathbf{B}_{4x1} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} \text{ and } \mathbf{C}_{2x2} = \begin{bmatrix} 1 & 0 \\ -7 & 4 \end{bmatrix}$$

Note that the row dimension always appears first. An element of a matrix may be identified by a double subscript, i.e.

a_{21}

would be the element in the second row and first column of **A**

i.e. $a_{21} = 2$, $a_{13} = -1$ and so on.

6.1 ADDITION OF MATRICES

Two matrices, say A and B, can be added *only* if they are of the same dimensions. The sum of two matrices will be the matrix obtained by adding *corresponding* elements of matrices **A** and **B**.

Example: Find **A** + **B** where

$$\mathbf{A}_{2x3} = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 0 \end{bmatrix} \mathbf{B}_{2x3} = \begin{bmatrix} 0 & -1 & 1 \\ 6 & -3 & 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 6 & -3 & 2 \end{bmatrix} = \begin{bmatrix} (2+0) & (1-1) & (4+1) \\ (-1+6) & (6-3) & (0+2) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 \\ 5 & 3 & 2 \end{bmatrix}$$

6.2 MULTIPLICATION OF A MATRIX BY A REAL NUMBER

We desire a rule for multiplying a matrix by a real number, i.e. **3A** where

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 3 & 6 \\ -1 & 6 \end{bmatrix}$$

$$\mathbf{3A} = \begin{bmatrix} 3(2) & 3(0) \\ 3(3) & 3(6) \\ 3(-1) & 3(6) \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 9 & 18 \\ -3 & 18 \end{bmatrix}$$

6.3 MATRIX MULTIPLICATION

The rule for matrix multiplication requires "row-column multiplication".

$$\mathbf{A}_{2x2} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \quad \mathbf{B}_{2x2} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$$

An element in the i^{th} row and j^{th} column of the product **AB** is obtained by multiplying the i^{th} row of **A** by the j^{th} column of B.

$$\mathbf{A}_{2x2}\,\mathbf{B}_{2x2} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 1 & 14 \end{bmatrix}$$

The first row - first column product would be

(2)(5) + (0)(-1) = 10

The first row - 2^{nd} column product

(2)(2) + (0)(3) = 4

The 2^{nd} row – 1^{st} column

(1)(5) + (4)(-1) = 1

The 2^{nd} row - 2^{nd} column

(1)(2) + (4)(3) = 14

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A}_{3x2}\ \mathbf{B}_{2x3} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 0 \\ 2 & -1 & -3 \\ 8 & 0 & 8 \end{bmatrix} = \mathbf{AB}_{3x3}$$

$$\mathbf{B}_{2x3}\ \mathbf{A}_{3x2} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 10 \end{bmatrix} = \mathbf{BA}_{2x2}$$

Note $\mathbf{A}_{mxp}\ \mathbf{B}_{pxq} = \mathbf{AB}_{mxq}$ the inner two dimensions must be equal.

6.4 IDENTITY ELEMENTS

In addition the identity element is 0, i.e.

0 + 2 = 2
0 + (-9) = -9

In multiplication the identity element is 1, i.e.

(1)(5) = 5
(1)(-4) = -4

In matrices the identity matrix for multiplication is matrix **I** where

AI = **A** and **IA** = **A**

The matrix called the *identity* matrix, is the *square* matrix

$$\mathbf{I}_{nxn} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

All elements in the main diagonal of the matrix are equal to 1.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix}$$

$$\mathbf{I}_{2x2}\ \mathbf{A}_{2x3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \mathbf{A}$$

$\mathbf{A}_{2x3}\ \mathbf{I}_{3x3} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 6 & 3 \end{bmatrix} = \mathbf{A}$

6.5 INVERSE OF A MATRIX

In order to solve the equation

$$2x = 6$$

multiply by 1/2, then $x = 3$, i.e. $2 \cdot 1/2 = 1$

Thus, the product of a number by its reciprocal must equal the identity element for multiplication. Let $\mathbf{A}_{nxn}$ be a square matrix, if a matrix $\mathbf{A}^{-1}$ can be found, such that

$\mathbf{AA}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

then $\mathbf{A}^{-1}$ is called the inverse of A.

6.6 TRANSPOSE OF A MATRIX

Let $\mathbf{A}_{pxq}$. Then **A'**, called the transpose of **A**, is defined to be a matrix obtained by interchanging corresponding rows and columns of A.

i.e. $\mathbf{A}_{3x2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 4 & 3 \end{bmatrix}$ *Then* $\mathbf{A'}_{2x3} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$

Ex 2 $\mathbf{Y}_{3x1} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ *Then* $\mathbf{Y'}_{1x3} = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}$

Note that $\mathbf{Y'Y} = \sum_{i=1}^{3} Y^2 i$

6.7 Exercises

1. Give the dimensions of the following matrices.

   a) $\begin{bmatrix} 2 & 6 \\ 3 & 1 \\ 1 & 4 \end{bmatrix}$

   b) $\begin{bmatrix} 2 & 4 & 7 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 4 \\ 1 & 9 & 8 \end{bmatrix}$

d) $\begin{bmatrix} x_1 & x_2 & \dots & x_9 \end{bmatrix}$

e) [6]

2. Identify the element in
   a) a_{11}
   b) b_{24}
   c) c_{33}

3. Which of the following are equal?

   a) $\begin{bmatrix} 2 & 1 \\ 0 & 6 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} \frac{4}{2} & \frac{2}{2} \\ \frac{0}{2} & \frac{12}{2} \\ \frac{4}{2} & \frac{8}{2} \end{bmatrix}$

   b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Give the transpose of the following matrices.
   a) $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

   b) $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

   c) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$

5. Add the following matrices.
   a) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

   b) $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 6 \\ 4 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

6. Multiply the following matrices.

a) $3\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 0 \\ 0 & 6 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 4 & 2 \\ 2 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

6.8 LINEAR EQUATIONS

The equation

$$\boxed{\boldsymbol{y = \beta_0 + \beta_1 X}}$$

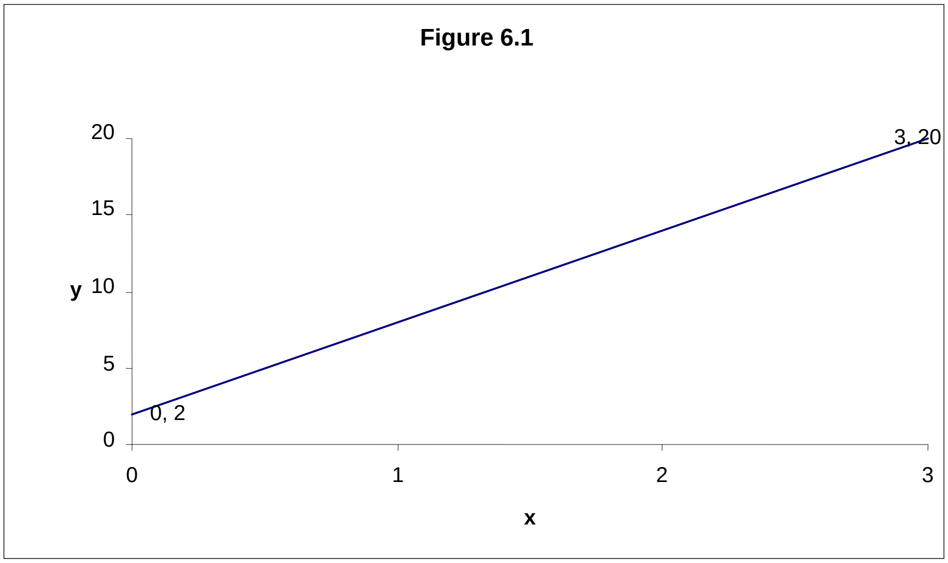
is the equation of a straight line. B_0 and β_1 are the parameters of the line. B_0 is the y-intercept, the value of y when $x = 0$ and β_1 is the slope of the line, how y changes per unit increase in x.

IF $y = 2 + 6x$

then when $x = 0$, $y = 2$. In other words, the y-intercept is 2.

IF x is increased by one unit (i.e. 1), then the value of y increases by 6. In other words, $\beta_1 = 6$.

A plot of the line can be easily produced. First plot the y-intercept (0, 2) and then select another point, say $x = 3$ then $y = 20$. Given that a straight line is determined by two points, connect the dots. See figure 6.1.



6.9 Exercises

1. Graph the following equations.
   a. $y = 2 + 3x$
   b. $y = 2 - 3x$
   c. $y = -2 + 3x$
   d. $R = 4 + 3V$

6.10 QUADRATIC EQUATIONS

The equation

$$\boxed{y = \beta^0 + \beta_1 X + \beta_2 X^2}$$

is the graph of a parabola. If $\beta_2 > 0$ the parabola opens upward $\cup$, whereas if $\beta_2 < 0$ the parabola opens downward $\cap$. The lowest (highest) point of the parabola when $\beta_2 > 0$ ($\beta_2 < 0$) is called a *verticie*. The value of X at this point is given by $\boldsymbol{X = -\beta_1 / 2\beta_2}$

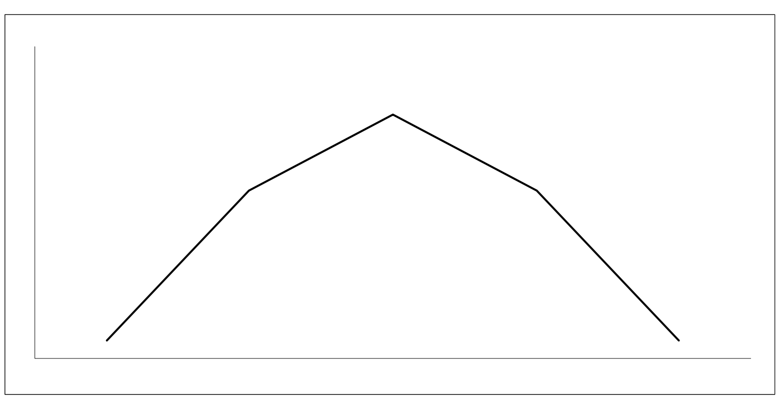


Figure 6.2

For example, consider $y = -8 - 2x + x^2$

Here, $\beta_0 = -8$, $\beta_1 = -2$, $\beta_2 = 1$, the vertex is $x = -\beta_1 / 2\beta_2 = -(-2) / 2*1 = 1$

$y = -8 - 2 * 1 + 1^2 = -9$

Vertex is (1, -9).

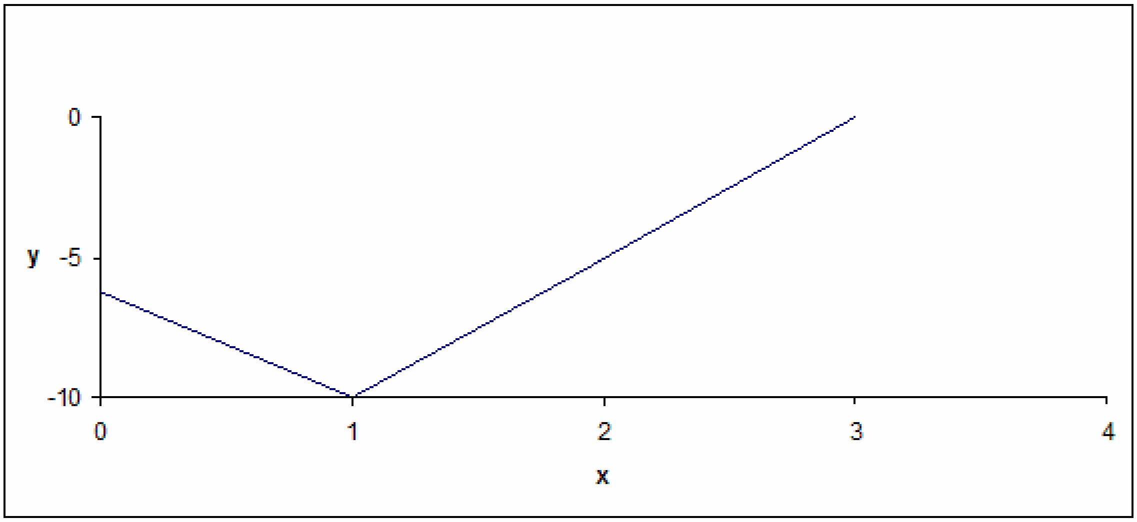


Figure 6.3

6.11 Exercises

1. Graph
   a. $y = -8 - 2x + 2x^2$
   b. $y = -4 + 4x - 2x^2$

6.12 POLYNOMIALS

The general polynomial of the kth degree in x is given by

$$\boxed{y = \beta_0 + \beta_1 X^1 + \beta_2 X^2 + \ldots + \beta_k X^k}$$

in which k is a positive integer and the β's are constants. The graph of a polynomial $y = f(x)$ can be obtained by computing several points (x, y) and joining them with a smooth curve.

- The *line* in section 6.7 can be called a *polynomial of degree one*.
- The *quadratic* in section 6.10 is also known as a *polynomial of degree two*.
- A *cubic* equation can be called a *polynomial of degree three*.

Psychologists rarely model systems with polynomial models of degree 3 or higher. Polynomials are used to model psychological systems, since in more advanced mathematics it can be shown that polynomials are smooth curves without breaks or sharp corners. This continuity models psychological systems very adequately.

Click here for some more examples.

6.13 Exercises

1. Graph $y = x^3 - 3x^2 - 3x + 5$ and let x range from -5 to +5.