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“A positive theory of income taxation where politicians focus upon swing and core voters<sup>\*</sup>”

by

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Abstract. We construct an equilibrium model of party competition, in which parties are especially concerned with their core and swing voters, concerns which American political scientists have focused upon in their attempts to understand party behavior in general elections.

Parties compete on a large policy space of possible income-tax policies. An element in this infinite-dimensional space is a function which maps pre-fisc income into post-fisc income. The only restrictions are that the function be continuous, and satisfy exogenously specified upper and lower bounds on its derivative, where it is differentiable. Only a fraction of each voter type will vote for each party, perhaps because of issues not modeled here or voter misperceptions of policies. Each party's policy makers comprise two factions, one concerned with maximizing the welfare of its constituency, or its *core*, the other with winning over *swing* voters. An equilibrium is a pair of parties (endogenously determined), and a pair of policies, one for each party, in which neither party can deviate to another policy which will be assented to by both its

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core and swing factions. Formally, this is a Nash equilibrium where each party possesses only a quasi-order over the policy space. We fully characterize the equilibria. There are many. In a specially important case, each party proposes a piece-wise linear tax schedule, and these schedules coincide for a possibly large interval of middle-income voters, while the ‘left’ party gives more to the poor and the ‘right’ party more to the rich.

An empirical section uses the data of Piketty and Saez on taxation in the US during the twentieth century to assess the model’s predictions. We argue that the model is roughly confirmed.

Key words: political economy, income taxation, political equilibrium

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The spirit of a people, its cultural level, its social structure, the deeds its policy may prepare—all this and more is written in its fiscal history, stripped of all phrases. He who knows how to listen to its message here discerns the thunder of world history more clearly than anywhere else.<sup>1</sup>

## 1. Introduction

Formal political-economic analysis of taxation has been in the main of a schematic nature: that is, existing models of income taxation usually assume that taxation is an affine function of income<sup>2</sup>. In reality, income-tax policy is extremely complex, reflecting the fact that many competing interests must be satisfied, or attended to. (For a useful ‘short history’ of the income tax in the United States, see Brownlee (2004), which contains a full guide to the literature.) In this paper, I attempt to capture this complexity by modeling political competition over the income tax as taking place on an infinite dimensional space of functions. Each political party will propose a function which will define the post-tax-and-transfer income, for every possible realization of pre-tax income, and these functions will be chosen from a large space, constrained only by upper and lower bounds on what the marginal tax rates can be<sup>3</sup>.

We will suppose that two parties are competing in a general election, and that the platform of each party consists in a proposal of such a ‘post-fisc’ income function. The paper’s positive aspect is to model the view that parties concentrate on *core* and *swing* voters, a view which is ubiquitous in contemporary American political science<sup>4</sup>. A simple way of formalizing these aims of a party is to assume that there are intra-party

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<sup>1</sup> Schumpeter (1954 [1918]), as quoted in Brownlee (2004).

<sup>2</sup> A number of papers, for example Dixit and Londregan (1998), study ‘pork barrel politics,’ in which parties propose payments to each of a finite number of voter types. Here, the policy space is finite dimensional, but could be of high dimension.

<sup>3</sup> I first studied taxation on this policy space in Roemer (2006). The present paper presents several new equilibrium concepts. The optimization techniques used here are similar to the ones employed in that monograph.

<sup>4</sup> See Cox (2006) for a recent review of the formal literature which attempts to model parties’ concerns with swing and core voters.

factions concerned, respectively, with these two problems – of satisfying the core constituency, and of appealing to the swing voters. We show that this solves the problem of the existence of a (Nash-type) equilibrium in pure strategies in the game of party competition, even when the parties are choosing strategies from an infinite dimensional space.

Besides modeling parties as complex organizations (in the sense that policy is set by intra-party bargaining, rather than by the maximization of a single payoff function), we depart from traditional formal approaches in the study of political competition in another way. The polity consists of a continuum of voter types, where ‘type’ is defined by the pre-tax income of the agent or household. In many – perhaps most -- formal papers about political competition, parties represent *no* constituencies. This is the case with the Downs model, where each party is only the vehicle of a candidate who seeks election. It is as well essentially the case with the citizen-candidate models of Osborne and Slivinski (1996) and Besley and Coate (1997), where candidates run on their own ideal policies (each ‘party’ represents a constituency of one type). In models where parties represent non-trivial constituencies – see for instance Dixit and Londregan (1998), Austen-Smith(2000), Levy (2004), and Roemer (1999, 2001,2006) -- it is supposed that each party represents an element of some partition of the polity. Here, we depart from these practices, by recognizing that, in reality, it is never the case that the sets of voters who support the various parties form an easily defined partition of the space of voter types. For instance, in American elections, a substantial fraction of voters *at every income level* supports each of the two parties: see figure 8 below for the details, and McCarty, Poole and Rosenthal (2006, chapter 3). Of course, one can say that, if the *space of types* were modeled as having sufficiently large dimensionality, there would always be characteristics of voters that would enable us to define the set voting for a particular party as an element of a partition of that space. We prefer, however, to take a more statistical approach – to keep the space of types of small dimension, and to say that, from the viewpoint of parties, there is a random element in voting, and therefore, if a party is concerned to represent its constituents, it has to attempt to represent every household type, at least to some extent – because *some* fraction of every type will vote for each party.

We will define an equilibrium concept reflecting these concerns, and then characterize an important class of equilibria in the income-tax competition game. The main characteristics of the equilibria are these:

1. In every equilibrium, there is a ‘Left’ and a ‘Right’ party. The Left party puts more weight on the interests of voters the poorer they are, and the Right puts more weight on the interests of voters, the richer they are;
2. In an important sub-class of these equilibria, each party proposes a piece-wise linear post-fisc policy;
3. In every such equilibrium, the policy proposed by Left entails an increasing average rate of taxation on the whole domain of incomes; the policy proposed by the Right entails an average rate of taxation that increases up to a point, and then decreases;
4. There is a two dimensional manifold of these equilibria, where a particular equilibrium can be viewed as being characterized by the relative strength of the ‘swing’ versus ‘core’ factions within each of the two parties;
5. In every equilibrium, the two parties propose exactly the same tax treatment for what may be a substantial interval of middle-income voters. The greater the focus the parties place upon swing voters, the larger will be the size of this interval.

In section 2, we propose several concepts of political equilibrium. In section 3, we characterize ‘left-right’ equilibria in the income-tax setting. Section 4 presents an alternative equilibrium concept, that might seem, a priori, to appeal. In section 5, we examine US income-tax data to see how well reality conforms to the model’s predictions. Section 6 discusses and concludes. Lengthy proofs are gathered in the appendix.

## 2. A concept of political equilibrium in two-party politics

Rather than present the equilibrium concepts for a general model, we specialize immediately to the case of income taxation.

### A. The policy space

A fiscal policy (or an income tax policy) is a mapping  $X : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which associates to any pre-tax income  $h$  a post-tax-and-transfer income. We assume that the

pre-tax income distribution is given by a cdf  $F$  on  $\mathbb{R}_+$ ; its mean is  $\mu$ . The policy space  $\mathfrak{S}$  consists of functions  $X : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that:

(P0)  $X$  is continuous,

(P1)  $0 \leq \alpha \leq X' \leq 1$ , some  $\alpha$ , where ever  $X$  is differentiable,

(P2)  $\int X(h)dF(h) = \mu$

where  $\alpha$  is a number,  $0 \leq \alpha < 1$ <sup>5</sup>. The two conditions (P1) and (P2) state that the derivative of  $X$ , where it exists, lies between  $\alpha$  and 1, and that  $X$  redistributes pre-tax income fully. (P0) is best justified as a condition of horizontal equity.

If the policy is  $X$ , then the net taxes paid by an individual  $h$  are  $t(h; X) = h - X(h)$ . Hence the marginal tax rate for  $h$  at policy  $X$  is  $1 - X'(h)$ , which is bounded below and above by zero and  $1 - \alpha$ , respectively. Leisure is not an argument of the utility function for reasons of tractability: the equilibrium analysis would otherwise become unmanageable. I attempt to recognize the elasticity of labor supply by requiring that the marginal tax rate be at most  $1 - \alpha$ : political parties agree not to consider policies that have very high marginal tax rates, because of the deleterious labor-supply effects ( $\alpha$  is a parameter of the model). Alternatively put, we are assuming that when marginal tax rates lie in the interval  $[0, 1 - \alpha]$ , labor-supply elasticity is very small and can be ignored<sup>6</sup>.

Obviously,  $\mathfrak{S}$  is a space of infinite dimension, so chosen to model the idea that political competition is ruthless, not being constrained by arbitrary mathematical constraints (such as linearity).

## B. Voter behavior

A voter's predicted utility at a policy  $X$  is her post-fisc income,  $X(h)$ .

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<sup>5</sup> More precisely, we can eliminate the requirement that  $X$  be continuous and replace the condition (P1) with a Lipschitz condition on  $X$ .

<sup>6</sup> Of course, it would be desirable to model labor-supply elasticity explicitly. In the models here, that would necessitate solving a 'double Mirrlees' optimal tax problem. Analytical simplicity would be lost.

However, voters actually behave stochastically. When voters face the two policies  $X^a$  and  $X^b$  from parties  $a$  and  $b$ , the share of voters of type (income)  $h$  that votes for  $X^a$  will be  $S(X^a(h), X^b(h))$  where  $S$  is a function with these properties:

$$(S1) \ S : \mathbb{R}_+^2 \rightarrow [0,1], \ S \text{ continuous}$$

$$(S2) \ S \text{ is non-decreasing in } x \text{ and non-increasing in } y$$

$$(S3) \ S(x, y) + S(y, x) = 1$$

Thus, if two policies  $X^a, X^b$  are competing, the fraction of the vote received by  $X^a$  will be

$$\sigma(X^a, X^b) = \int S(X^a(h), X^b(h)) dF(h). \quad (2.1)$$

Condition (S3) implies that  $S(x, x) = 0.5$ , so it is assumed that no income *type* is biased towards one party. This assumption can be weakened, but we choose to keep the model as simple as possible.

Example 1. Suppose that a particular voter of type  $h$  votes for  $X^a$  when  $X^a(h) > \delta X^b(h)$ , and among voters of type  $h$ , the random variable  $\delta$  is distributed according to a distribution function  $G$  on  $\mathbb{R}_+$  such that for all  $x \in (0,1]$ ,  $G(x) + G(1/x) = 1$ . Then the share of type- $h$  voters voting for  $X^a$  will be  $S(X^a(h), X^b(h)) = G\left(\frac{X^a(h)}{X^b(h)}\right)$ . Conditions (S1)-(S3) are satisfied. It is easy to generate such distribution functions.

Example 2. Let  $S(x, y) = \frac{x}{x+y}$ . Conditions (S1)-(S3) hold. This function is useful for computing examples, although it lacks a stochastic micro-foundation.

### C. Political equilibrium

We propose a concept of political equilibrium in which parties are endogenous, and each party contains political entrepreneurs who adopt different strategies. One strategy is to attempt to represent the *constituency* of the party; the other strategy is to target *swing voters*. The constituency of the party and the swing voters are endogenous.

We propose to define the core (constituency) of a party as an *historical* and *statistical* concept. Suppose in the last election, at date  $t - 1$ , the set of voters who voted for the party is characterized by the function  $\theta_{t-1}$  defined by

$$\theta_{t-1}(h) = S(X_{t-1}^a(h), X_{t-1}^b(h)).$$

The party at date  $t$  identifies its *core* as the voters so described: that is as a fraction  $\theta_{t-1}(h)$  of voters of type  $h$ , for every  $h$ .

We say the *swing voters* comprise the set of income types  $\{h \mid \theta(h) = \frac{1}{2}\}$ .

We now discuss the behavior of political entrepreneurs, who set policy for the parties. We assume there are two parties. Parties exist for a long time; they build a reputation by representing certain constituencies. With stochastic voting, the constituency of a party is hard to define, because one can never be sure exactly who will vote for the party. Nevertheless, from a statistical viewpoint, the constituency of a party may be quite clear, as I have indicated.

We suppose at the present election (date  $t$ ) those politicians who attempt to represent the party's constituency want to choose the policy  $X$  to maximize the function:

$$\int \theta_{t-1}(h)X(h)dF(h).$$

That is, they will attempt to maximize the average welfare of their statistical constituency, by weighting the welfare of every income type by the fraction of that type that comprise the constituency of the party<sup>7</sup>. Formally, they desire to represent every income type, but with varying weights. We depart from the more familiar formulation that each party represents a distinct set of voter types.

We model the second faction of politicians, the 'swing faction,' as insisting that the party promise at least as much to the swing voters as the other party is proposing to give them. They are battling for the loyalty of the swing voters.

We first propose a concept of a sequence of political equilibria over time. Note, from the above, that the party's constituency is defined by the last election. We suppose

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<sup>7</sup> Some aggregation principle (i.e., social welfare function) other than summing up could be used. The key point is that types are weighted by their historical loyalty to the party.



that the distribution of types,  $F$ , is unchanging over time (i.e., the distribution of types changes slowly compared to the period of the election cycle).

Definition 1 A history of political equilibria given a function  $\theta_0 : H \rightarrow [0,1]$  is a sequence of policies  $\{(X_t^L, X_t^R) \in \mathfrak{S} \times \mathfrak{S} \mid t = 1, 2, \dots\}$ , and a sequence of functions  $\{\theta_t : H \rightarrow [0,1] \mid t = 1, 2, \dots\}$  such that:

(1a) for every  $t=1, 2, \dots$  policy  $X_t^L$  solves the following program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int \theta_{t-1}(h) X(h) dF(h) \\ & \text{subj. to } (\forall h) (\theta_{t-1}(h) = \frac{1}{2} \Rightarrow X(h) \geq X_t^R(h) \quad (L1) \end{aligned}$$

(1b) for every  $t=1, 2, \dots$ , policy  $X_t^R$  solves the program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int (1 - \theta_{t-1}(h)) X(h) dF(h) \\ & \text{subj. to } (\forall h) (\theta_{t-1}(h) = \frac{1}{2} \Rightarrow X(h) \geq X_t^L(h) \quad (R2) \end{aligned}$$

(2) for every  $t=1, 2, \dots$ , and for all  $h \in H$  :

$$\theta_t(h) = S(X_t^L(h), X_t^R(h)).$$

From (2), the function  $\theta_t$  gives the fraction of each type that votes for party  $L$  in the election at date  $t$ . The constraints (L1) and (R1) in the two programs are imposed by the factions concerned with swing voters: for instance, (L1) says that party  $L$  can propose no policy that provides lower utility to the swing voters than the  $R$  party proposes to provide them. In words, a history of political equilibria comprises a sequence of pairs of policies such that, given the party's conception of its statistical constituency from the election held at date  $t - 1$ , the policy of the party at date  $t$  cannot be dominated by any other policy with respect to weighted average welfare of the party's historical constituency, subject to providing at least as well as the opposition proposes to provide for swing voters.

The datum of the equilibrium concept is the pair of functions  $(F, \theta_0)$ . We may view  $\theta_0$  as the initial conjecture of the two parties concerning their statistical constituencies.

One can ask: Why not model the ‘swing voter faction’ as maximizing vote share? We will study this alternative in section 4.

We can expect that there will be many possible histories of political equilibria. If one is interested in modeling general elections to understand the *underlying long-range* political conflicts in a society, then one should be interested in stationary points of these histories. An interest in stationary points must, of course, be justified by a view that the underlying distribution of preferences, represented by  $F$ , is changing slowly relative to the frequency of elections.

I propose a concept of stationarity which entails that the sequence of functions  $\{\theta_t\}$  in a history of political equilibria converges to a function  $\theta_*$ : thus, the constituency of each party becomes stable.

**Definition 2** A *stationary equilibrium* is a function  $\theta_* : \mathbb{R}_+ \rightarrow [0,1]$  and a pair of policies  $(X_*^L, X_*^R)$  such that:

( $\alpha$ 1a) policy  $X_*^L$  solves the program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int \theta_*(h) X(h) dF(h) \\ & \text{subj. to } (\forall h) (\theta_*(h) = \frac{1}{2} \Rightarrow X(h) \geq X_*^R(h) \quad (L2) \end{aligned}$$

( $\alpha$ 1b) policy  $X_*^R$  solves the program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int (1 - \theta_*(h)) v(X; h) dF(h) \\ & \text{subj. to } (\forall h) (\theta_*(h) = \frac{1}{2} \Rightarrow X(h) \geq X_*^L(h) \quad (R2) \end{aligned}$$

( $\alpha$ 2) For all  $h \in H$ ,  $\theta_*(h) = S(X_*^L(h), X_*^R(h))$ .

There is a kind of equilibrium which is a refinement of stationary equilibrium and plays an important role in the analysis:

**Definition 3** A *1-stationary equilibrium* is a function  $\theta_*$ , a pair of policies  $(X_*^L, X_*^R)$ , and an ordered pair  $(h_*, y) \in \mathbb{R}_+^2$  such that:

( $\beta 1a$ )  $X_*^L$  solves the program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int \theta_*(h) X(h) dF(h) \\ & \text{subj. to } X(h_*) \geq y \quad (L3) \end{aligned}$$

( $\beta 2a$ )  $X_*^R$  solves the program

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int (1 - \theta_*(h)) X(h) dF(h) \\ & \text{subj. to } X(h_*) \geq y \quad (R3) \end{aligned}$$

( $\beta 3$ ) For all  $h \in H$ ,  $\theta_*(h) = S(X_*^L(h), X_*^R(h))$

( $\beta 4$ )  $X_*^L(h_*) = y = X_*^R(h_*)$ .

In this concept, it is as if the vote-share-seeking faction is concentrating on not losing the loyalty of one swing voter type, namely  $h^*$ . What is important is the relationship of 1-stationary equilibrium to stationary equilibrium.

Proposition 1 *Every 1-stationary equilibrium is a stationary equilibrium.*

Proof:

Let  $(\theta_*, X_*^L, X_*^R, h_*, y)$  be a 1-stationary equilibrium. By ( $\beta 4$ ), we can write the constraint ( $L3$ ) as  $X(h_*) \geq X_*^R(h_*)$ . But by ( $\beta 4$ ), it also follows that  $\theta_*(h_*) = \frac{1}{2}$ . Therefore, constraint ( $L3$ ) is weaker than constraint ( $L2$ ). Hence the program in ( $\beta 1a$ ) has the same objective function but a larger opportunity set than the program in ( $\alpha 1a$ ). However,  $X_*^L$  is a member of the opportunity set defined by ( $L2$ ). It follows that  $X_*^L$  solves ( $\alpha 1a$ ). In like manner,  $X_*^R$  solves ( $\alpha 1b$ ), proving the claim. ■

Now examine the program in condition ( $\beta 1a$ ). It is a concave programming problem. There is no interaction between the  $L$  and  $R$  policies – so in principle it can be solved. The same goes for the program in ( $\beta 1b$ ). In other words, the refined equilibrium concept of 1-stationary equilibrium is quite tractable.

I will proceed by applying these equilibrium concepts to the study of income taxation, and will then argue, by looking at the history of income tax reform, that the equilibrium concept appears to explain quite well some of its main features.

### 3. Analysis

The first theorem will characterize a two dimensional family of 1-stationary equilibria. To do so, we define two families of piece-wise linear functions. Fix a number  $h_* > 0$ . The first family is

$$M_a(h_*) = \left\{ X \in \mathfrak{S} \mid \exists (x_a, h_1) \in \mathbb{R}_+^2 \text{ such that } h_1 \leq h_* \text{ and } \right. \\ \left. X(h) = \begin{cases} x_a + \alpha h, & \text{if } h \leq h_1 \\ x_a + \alpha h_1 + (h - h_1), & \text{if } h_1 < h \leq h_* \\ x_a + \alpha h_1 + (h_* - h_1) + \alpha(h - h_*), & \text{if } h > h_* \end{cases} \right\}.$$

A typical function in the family is graphed in figure 1.  $M_a(h_*)$  is a unidimensional family of functions which we may view as being parameterized by the value  $y \equiv X(h_*)$ ; that is, fixing the ordered pair  $(h_*, y)$  determines at most one policy in the family  $M_a(h_*)$ . By construction, the policies  $X \in M_a(h_*)$  satisfy (P0) and (P1). The budget- balance condition (P2) gives one equation in the two unknowns  $(x_a, h_1)$ : hence, the unidimensionality of this family.

The second family is:

$$M_b(h_*) = \left\{ X \in \mathfrak{S} \mid \exists (x_b, h_2) \in \mathbb{R}_+^2 \text{ such that } h_2 \geq h_* \text{ and } \right. \\ \left. X(h) = \begin{cases} x_b + h, & \text{if } h \leq h_* \\ x_b + h_* + \alpha(h - h_*), & \text{if } h_* < h \leq h_2 \\ x_b + h_* + \alpha(h_2 - h_*) + (h - h_2), & \text{if } h > h_2 \end{cases} \right\}.$$

Likewise,  $M_b(h_*)$  is a unidimensional family of piece-wise linear policies, which is parameterized by  $y \equiv X(h_*)$ ; a typical policy is also graphed in figure 1.

We will be interested in policy pairs  $(X^a, X^b) \in M_a(h_*) \times M_b(h_*)$  which share a common value of  $y = X^a(h_*) = X^b(h_*)$ . The next proposition tells us exactly what the admissible range is for  $y$ .

**Proposition 2** *Let  $h_* > 0$ , and let  $y$  lie in the interval*

$$\max[(1 - \alpha)\mu + \alpha h_*, h_*] \leq y \leq h_* + (1 - \alpha) \int_{h_*}^{\infty} (h - h_*) dF(h). \quad (3.1)$$

*Then:*

A. *There exist unique policies  $X_a \in M_a(h_*)$ ,  $X_b \in M_b(h_*)$  such that*

$$X_a(h_*) = y = X_b(h_*). \quad (3.2)$$

B. Conversely, if  $y$  does not lie in the interval defined by (3.1), then there is no pair of policies in the two families for which (3.2) holds.

C. The number  $x_a$  is positive, and the number  $x_b$  is non-negative, and positive except in a singular case.

All longer proofs, beginning with the proof of this proposition, appear in the Appendix.

Define:

$$\Gamma = \{(h_*, y) \in \mathbb{R}_+^2 \mid \max[(1-\alpha)\mu + \alpha h_*, h_*] \leq y \leq h_* + (1-\alpha) \int_{h_*}^{\infty} (h - h_*) dF(h)\}$$

and

$$y_{\min}(h_*) = \max[(1-\alpha)\mu + \alpha h_*, h_*], \quad y_{\max}(h_*) = h_* + (1-\alpha) \int_{h_*}^{\infty} (h - h_*) dF(h).$$

Proposition 2 tells us that for any  $(h_*, y) \in \Gamma$ , there exists a unique pair of policies

$X^a \in M_a(h_*)$  and  $X^b \in M_b(h_*)$  such that

$$X^a(h_*) = y = X^b(h_*).$$

To avoid notational complexity, let us fix  $(h_*, y) \in \Gamma$  and denote these two functions simply by  $X^a$  and  $X^b$ . Figure 1 displays the graphs of a typical pair of such functions. Note, in particular, that these two policies coincide on the interval  $[h_1, h_2]$ . Suppose, now, that these two policies are being proposed by the parties,  $a$  and  $b$ , and define the function  $\theta(\cdot; X^a, X^b)$  by equation (2.1). We have:

**Proposition 3.** *When  $X^a \neq X^b$ , the function  $\theta(\cdot; X^a, X^b)$  is decreasing on the interval  $[0, h_1]$ , constant and equal to one-half on the interval  $[h_1, h_2]$ , and decreasing on the interval  $(h_2, \infty)$ .*

**Proof:**

Easily verified from the definition of the functions  $X^a$ ,  $X^b$ , and  $S$ . ■

We may now state our first main result:

**Theorem 1** *Let  $(h_*, y) \in \Gamma$ , and let  $X^a \in M_a(h_*)$ ,  $X^b \in M_b(h_*)$  such that*

$X^a(h_*) = y = X^b(h_*)$ . Let  $\theta(\cdot; X^a, X^b)$  be defined as above. Then  $(\theta, X^a, X^b)$  is a 1-stationary equilibrium.

Theorem 1 gives us a stationary equilibrium for each  $(h_*, y) \in \Gamma$ : thus, a two-parameter family of equilibria. We propose an interpretation of the political nature of these various equilibria, which follows from the next result. In Figure 2, we graph the manifold  $\Gamma$  for the case where  $F$  is the lognormal distribution of income with mean 50 and median 40, an approximation of the US household income distribution in 2000, in units of \$1000, and  $S(x, y) = \frac{x}{x+y}$ . Note that for  $h_* \geq \mu$ , the lower envelope of  $\Gamma$  coincides with the  $45^\circ$  ray.

**Theorem 2** A. Consider a point  $(h_*, y_{\max}(h_*))$  on the upper envelope of the manifold  $\Gamma$ . Let the two policies of the 1-stationary equilibrium at this point be denoted  $X^L$  and  $X^R$ . Then  $h_1 = 0$ ,  $h_2 = \infty$  and  $X^L = X^R = X^*$ , where  $X^*$  is the ideal policy in  $\mathfrak{S}$  of voter  $h_*$ .  
 B. Consider a point  $(h_*, y_{\min}(h_*))$  on the lower envelope of  $\Gamma$ , with its associated 1-stationary equilibrium  $(X^L, X^R)$ . If  $h_* \leq \mu$  then  $X^L$  is the ideal policy in  $\mathfrak{S}$  of Left's constituency<sup>8</sup>, and if  $h_* \geq \mu$  then  $X^R$  is the ideal policy in  $\mathfrak{S}$  of Right's constituency.

Fix the 'pivot type'  $h_*$  and begin at the equilibrium on the upper envelope of  $\Gamma$  at  $h_*$ . In the stationary equilibrium at this point, both parties propose the ideal policy of the pivot type,  $h_*$ . Each party receives half the vote. Here we have politics where the concern for swing voters is very strong in both parties: the factions representing constituent interests have no pull. As we start to move vertically down the manifold  $\Gamma$ , decreasing  $y$  and holding  $h_*$  fixed, the two policies diverge. The factions concerned with core voters become more powerful in intra-party bargaining. When we reach the equilibrium on the lower envelope of  $\Gamma$ , if  $h_* < \mu$ , then this faction is entirely dominant in the Left party in the sense that the L party is playing as if it is only concerned with constituent interests; if  $h_* > \mu$ , then constituent interests are dictating policy in the Right

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<sup>8</sup> That is,  $X^L$  maximizes  $\int \theta(h)X(h)dF(h)$ , for  $X \in \mathfrak{S}$ , where  $\theta(h) \equiv \theta(h; X^L, X^R)$ .

party. In the singular case that  $h_* = \mu$ , both parties are maximizing over  $\mathfrak{S}$  the average utility of their statistical constituencies.

For a policy  $X$ , define the average tax rate at  $h$  as:

$$a(h; X) = \frac{h - X(h)}{h}.$$

Define a policy as progressive if it unambiguously redistributes from the rich to the poor, in the following sense:

**Definition 4** A policy  $X$  is *progressive* if there exists  $\hat{h}$  such that:

$$h \leq \hat{h} \Rightarrow X(h) \geq h$$

$$h > \hat{h} \Rightarrow X(h) \leq h$$

and at least one (some) of these inequalities hold(s) strictly for some  $h$ .

We have:

**Proposition 4** Consider any 1-stationary equilibrium  $(X^L, X^R)$  of theorem 1. Then:

- A.  $a(\cdot; X^L)$  is increasing on  $\mathbb{R}_+$ , and  $a(\cdot; X^R)$  is increasing on  $[0, h_2)$  and decreasing on  $(h_2, \infty)$ , except in the singular case that  $X^R$  is the *laissez-faire* policy.
- B. Both policies are progressive, except in the singular case that  $X^R$  is the *laissez-faire* policy.

**Proof:**

A. The condition  $\frac{d}{dh}a(h; X) > 0$  is equivalent to  $hX'(h) < X(h)$ . It is easy to check (for instance, examine Figure 1) that this condition is true for  $X^L$ : formally, this follows from the fact that  $x_a > 0$  and  $x_b \geq 0$  (see Prop. 2(c)). For  $X^R$ , it is also easy to check that the condition holds if and only if  $h < h_2$ . Now note that if the segment of the graph of  $X^R$  on the domain  $[h_2, \infty)$  is extended into a line, it passes below the origin. (For if it passed through or above the origin, the policy  $X^R$  would dominate the policy  $X(h) \equiv h$ , and would not be feasible, as it would integrate to more than  $\mu$ .) This means that  $hX' > X$  on  $[h_2, \infty)$ .

B. This follows immediately from the fact that  $x_a > 0$  and  $x_b > 0$  (except in the laissez-faire policy)<sup>9</sup>. ■

We provide some simulations showing these stationary equilibria. We choose  $F$  to be the lognormal distribution with mean 50 and median 40. In figure 3, we graph the 1-stationary equilibrium of theorem 1 for four values of  $y$ , holding  $h_*$  at the mode of the income distribution. When  $y = y_{\max}(h_*)$  (figure 3a), the two policies coincide at the ideal policy of  $h_*$ ; when  $y = y_{\min}(h_*)$  (figure 3c), Left is playing the ideal policy of its average constituency, since  $h_* < \mu$ . Even on the lower envelope of the manifold, the policies agree for 16% of the polity.

In Figure 4, we plot the average tax rate functions for the Left and Right policies, at a point in  $\Gamma$ . The Right imposes a higher average tax rate up to  $h_1$ ; of course average tax rates of the two policies coincide on the interval  $[h_1, h_2]$ ; the Left policy imposes a higher average tax rate on  $(h_2, \infty)$ . Moreover, the Left average tax rate is monotone increasing on the whole domain, while the Right average tax rate is increasing until  $h_2$ , and then monotone decreasing asymptotically to zero thereafter.

We next ask what is the effect of a change in  $h_*$  on equilibrium policies. We graph some examples to show the contrast. In figure 5, we present the average tax rate functions associated with Left and Right policies for two values,  $h_* \in \{20, 80\}$ . In each case we plot policies for  $(h_*, \frac{y_{\min}(h_*) + y_{\max}(h_*)}{2}) \in \Gamma$ . We see that the effect of increasing the ‘pivot’  $h_*$  is to flatten out the average tax rate functions. For large values of the pivot, both parties propose net tax rates of close to zero for middle-income voters. Further discussion of how to interpret the 2-manifold of equilibria appears in section 5.

We next note the central role of 1-stationary equilibria in the theory.

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<sup>9</sup> Indeed a stronger statement can be made: Every policy in  $\mathfrak{S}$  except the laissez-faire policy is progressive. This is simply a consequence of the fact that the graph of a policy can only cross the 45° ray once.



Definition 5 A *left-right stationary equilibrium* is a stationary equilibrium where the function  $\theta_*$  is weakly monotone decreasing,  $\theta_*(0) \geq \frac{1}{2}$ , and for sufficiently large  $h$ ,

$$\theta_*(h) \leq \frac{1}{2}.$$

In other words, such an equilibrium is one where one party (Left) gives more weight to voters the poorer they are, the other (Right) gives more weight to voters, the richer they are. (We include the case where  $\theta_*(h) \equiv 1/2$  as an instance of ‘Left-Right’ equilibrium only for semantic convenience, to simplify the statement of theorems.) The next theorem tells us how 1-stationary equilibria come about historically:

Theorem 3. *Suppose that  $\theta_0(\cdot)$  is a weakly monotone decreasing function, and there is a*

*unique income  $h_*$  such that  $\theta_0(h_*) = \frac{1}{2}$ . Let  $y \in [y_{\min}(h_*), y_{\max}(h_*)]$  and let*

*$(\hat{X}^L, \hat{X}^R) \in M_a(h_*) \times M_b(h_*)$  be the unique policies associated with the ordered pair*

*$(h_*, y) \in \Gamma$ . Then  $(\hat{X}^L, \hat{X}^R)$  is a stationary equilibrium reached at date 1 beginning*

*from  $\theta_0$  in a sequence of historical equilibria. Conversely, let  $(X^L, X^R)$  be any*

*equilibrium reached at date 1 beginning from  $\theta_0$ , in a sequence of historical equilibria.*

*Then  $(X^L, X^R)$  are precisely the policies in  $M_a(h_*) \times M_b(h_*)$  associated with the ordered pair  $(h_*, X^L(h_*)) \in \Gamma$ .*

If we assume, in a history of political equilibria, that once a stationary equilibrium is reached, it continues to be played at all future dates, then theorem 3 says that all sequences of historical equilibria which begin with a vote share function  $\theta_0$  as specified in the premise end in one period, with a left-right 1-stationary equilibrium, as depicted in figure 1.

We next will describe the stationary equilibria reached by sequences of historical equilibria associated with historical share functions  $\theta_0$  which are weakly monotone

decreasing and for which there is a non-degenerate interval  $[h_*, h_{**}]$  upon which  $\theta_0 = \frac{1}{2}$ .

Let  $Z : [h_*, h_{**}] \rightarrow \mathbb{R}_+$  be an arbitrary, continuous function such that  $\alpha \leq Z'(h) \leq 1$ . If the integral  $(dF)$  of  $Z$  on  $[h_*, h_{**}]$  is neither too small nor too large, then there exists a left-right stationary equilibrium, in which both L and R policies coincide with  $Z$  on the interval  $[h_*, h_{**}]$ , and are as depicted in Figure 6a. On the intervals  $[0, h_*]$  and  $[h_{**}, \infty)$  the two policies behave just as the policies in a 1-equilibrium (see figure 1). Note that if  $h_{**} = h_*$ , then figure 6a becomes exactly figure 1.

To be precise:

**Theorem 4.** Let  $\theta_0$  be weakly monotone decreasing such that  $\theta_0(h) = \frac{1}{2}$  on  $[h_*, h_{**}]$ . Let

$Z$  be a continuous function defined on  $[h_*, h_{**}]$  such that

$$(\forall h \in [h_*, h_{**}])(\alpha \leq Z'(h) \leq 1).$$

Suppose that there exist numbers  $h_1 \in [0, h_*]$  and  $h_2 \in [h_{**}, \infty)$  and  $x_a \geq 0, x_b \geq 0$  such that the functions  $\hat{X}^L$  and  $\hat{X}^R$ , defined below, are continuous and integrate  $(dF)$  to  $\mu$ :

$$\hat{X}^L(h) = \begin{cases} x_a + \alpha h, & 0 \leq h \leq h_1 \\ x_a + \alpha h_1 + (h - h_1), & h_1 < h \leq h_* \\ Z(h), & h_* < h \leq h_{**} \\ Z(h_{**}) + \alpha(h - h_{**}), & h > h_{**} \end{cases}$$

$$\hat{X}^R(h) = \begin{cases} x_b + h, & 0 \leq h \leq h_* \\ Z(h), & h_* \leq h \leq h_{**} \\ Z(h_{**}) + \alpha(h - h_{**}), & h_{**} \leq h \leq h_2 \\ Z(h_{**}) + \alpha(h_2 - h_{**}) + h - h_2, & h > h_2 \end{cases}$$

Then  $(\hat{X}^L, \hat{X}^R)$  is a left-right stationary equilibrium, reached in one date from the historical vote share function  $\theta_0$ . Conversely, let  $(X^L, X^R)$  be any equilibrium beginning at  $\theta_0$  which is reached at the first date, and let  $Z(h) \equiv X^L(h)$  on  $[h_*, h_{**}]$ . Then the functions  $(\hat{X}^L, \hat{X}^R)$  can be defined as in the statement, they integrate  $(dF)$  to  $\mu$ , and  $(X^L, X^R) = (\hat{X}^L, \hat{X}^R)$ .

Theorems 1 and 4 characterize left-right stationary equilibria. Obviously, the 1-stationary equilibria are the simplest; there is a 2-manifold of them. The manifold of

equilibria of the form described in theorem 4 is infinite dimensional, for the function  $Z$  can be specified in an essentially arbitrary way.

We can locate a set of equilibria which lie ‘between’ the 1-equilibria and the equilibria of theorem 4. Let us define the concept of a 2-stationary equilibrium:

**Definition 6** A 2-stationary equilibrium is a tuple  $(h_*, y_*, h_{**}, y_{**})$  and a triple of functions  $(\theta_*, X^L, X^R)$  such that:

( $\gamma 1a$ )  $X^L$  solves

$$\max_{X \in \mathfrak{S}} \int \theta_*(h) X(h) dF(h)$$

s.t.

$$X(h_*) \geq y_*$$

$$X(h_{**}) \geq y_{**}$$

( $\gamma 2a$ )  $X^R$  solves

$$\max_{X \in \mathfrak{S}} \int (1 - \theta_*(h)) X(h) dF(h)$$

s.t.

$$X(h_*) \geq y_*$$

$$X(h_{**}) \geq y_{**}$$

( $\gamma 3$ )  $\theta_*(h) = S(X^L(h), X^R(h))$

( $\gamma 4$ )  $X^L(h_*) = y_* = X^R(h_*)$  and  $X^L(h_{**}) = y_{**} = X^R(h_{**})$ .

It will not surprise the reader that there is a 4-manifold of 2-stationary equilibria, parameterized by the choice of the vector  $(h_*, y_*, h_{**}, y_{**})$ ; they comprise piece-wise linear policies, where each policy has five pieces. A typical one is depicted in Figure 6b. The slopes of the line segments of each policy alternate between  $\alpha$  and 1, beginning with slope  $\alpha$  for the L policy and slope 1 for the R policy. Each of these is, of course, an equilibrium of the type described in theorem 4, where the function  $Z$  is a piece-wise linear function with two pieces. The theorem characterizing 2-stationary equilibria is again proved by the same method as theorem 1.

It is hard to imagine how the general left-right stationary equilibria of figure 6a might come about. In contrast, 1-stationary equilibria are easy to imagine, where the factions concerned with swing voters in each party concentrate on one voter type  $h_*$  – perhaps the mode of the income distribution (that’s where the votes are, as Willie Sutton might have said<sup>10</sup>). Even 2-stationary equilibria are imaginable, where the swing-voter factions are concentrating not on all the swing voters, but on two prominent income types. We can easily generalize this concept to  $n$ -stationary equilibrium, where the swing factions concentrate on not losing the loyalty of  $n$  voter types: this generates stationary equilibria where the policies are each piece-wise linear with  $2n + 1$  pieces. Thus, in the Eisenhower administration, when the piece-wise linear income-tax schedule in the United States had 17 pieces, we can imagine that eight income types had sufficient clout to convince both parties that their votes were up for grabs.

#### 4. An alternative formulation

One might object to the formulation of the swing voter faction: Why does this faction not attempt to guarantee that the party win at least half the votes? In this section, we study this alternative, and argue, in conclusion, that the model analyzed in section 3 may be a superior formulation, from the empirical viewpoint.

We now append two more requirements on the share function  $S$ . Let

$S_1(x, y) = \frac{\partial S}{\partial x}(x, y)$ . As well as satisfying conditions (S1)-(S3), we require:

(S4)  $S_1(x, x)$  is non-increasing in  $x$ ;

(S5)  $S(x, y)$  is concave, but not linear, in  $x$ .

Example 2 of section 2 satisfies (S1)-(S5). For example 1 of section 2, it is easy to construct distribution functions  $G$  such that (S1) through (S5) are satisfied. Let  $q: [0, 1] \rightarrow \mathbb{R}_+$  be any increasing differentiable concave function such that  $q(0) = 0$  and  $q(1) = 1/2$  and define

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<sup>10</sup> For non-American readers, the famous bank robber of the 1930s replied to the query why he robbed banks, “Because that’s where the money is.”

$$G(x) = \begin{cases} q(x), & 0 \leq x \leq 1 \\ 1 - q(1/x), & x > 1 \end{cases}$$

(For instance, take  $q(x) = (.5 - b)x^2 + bx$  for any number  $.5 \leq b \leq 1$ .) It is easy to check that (S1)-(S5) are satisfied when we define  $S(x, y) = G(x/y)$ .

**Definition 7** A triple  $(\theta, X^a, X^b)$  is an *alternative equilibrium* if:

(1) there is no policy  $X \in \mathfrak{S}$  such that:

$$\int \theta(h)X(h)dF(h) \geq \int \theta(h)X^a(h)dF(h), \text{ and}$$

$$\int S(X(h), X^b(h))dF(h) \geq \int S(X^a(h), X^b(h))dF(h) \geq \frac{1}{2}$$

with at least one of the first two inequalities strict;

(2) there is no policy  $X \in \mathfrak{S}$  such that:

$$\int (1 - \theta(h))X(h)dF(h) \geq \int (1 - \theta(h))X^b(h)dF(h), \text{ and}$$

$$\int S(X(h), X^a(h))dF(h) \geq \int S(X^b(h), X^a(h))dF(h) \geq \frac{1}{2}$$

with at least one of the first two inequalities strict;

(3)  $\theta(h) = S(X^a(h), X^b(h))$

This is a *refinement* of the PUNE concept, proposed by this author (Roemer [1999,2001]). In PUNE, there are two factions in each party, one of which (the Guardians) maximizes the average welfare of the constituency, and the other of which (the Opportunists) maximizes vote share. A PUNE satisfies conditions (1) and (2) but amended not to require that each party win at least half the vote. Clearly both parties must win exactly half the vote in an alternative equilibrium.

**Definition 8** A *strict Condorcet winner* is a policy that defeats all policies other than itself.

**Theorem 5** Let the share function  $S$  satisfy (S1) – (S5). Then:

A. The policy

$$\hat{X}(h) = (1 - \alpha)\mu + \alpha h$$

is a strict Condorcet winner in  $\mathfrak{S}$ .

B. If  $\theta(\cdot)$  is a (weakly) decreasing function, then  $\hat{X}$  maximizes  $\int \theta(h)X(h)dF(h)$  over  $X \in \mathfrak{S}$ .

C. The unique Left-Right alternative equilibrium<sup>11</sup> consists in both parties playing  $\hat{X}$ .

It is perhaps surprising that a Condorcet winner exists on this infinite dimensional space. Indeed, statement A of the theorem is remarkable, because it is true for any distribution  $F$  of types. (One might have thought that the highly redistributive policy  $\hat{X}$  would only be a Condorcet winner if  $F$  were sufficiently left-skewed.) This is, of course, a consequence of the definition of the policy space and the share function  $S$ . In particular, the proof invokes (S4) and (S5).

Statements B and C say that, in the only Left-Right alternative equilibrium, both parties play the ideal policy of the Left. In particular,  $\theta \equiv 1/2$ . This is not what we observe (see the next section); hence the alternative equilibrium appears not to be the right description of what parties are doing in the US.

Perhaps this adds credibility to the formulation in which parties have factions that attempt to represent the core and attract for swing voters (our ‘positive’ approach). Alternatively, one might attempt to develop a model with the alternative equilibrium, but weakening the conditions on the share function, so that  $\hat{X}$  would not be a strict Condorcet winner, thus vitiating the results of theorem 5. A caveat: assumption (S5) makes our optimization problem concave. If the function  $S(\cdot; y)$  were not concave, then an faction interested in maximizing vote share would be faced with a non-concave optimization problem on an infinite dimensional space, a very difficult problem.

## 5. Income tax rates in the United States

In this section, we ask how well the model performs in light of the recent historical record in the United States. We use the data on income taxation assembled by Piketty and Saez (2006). In their research, Piketty and Saez have used the public micro-file tax data of the IRS, and have computed the sum of four federal taxes for US taxpayers: the income tax, the social security and medicare payroll taxes, the estate tax,

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<sup>11</sup> That is, an alternative equilibrium in which  $\theta$  is weakly monotone decreasing.

and the corporate tax. The corporate tax is allocated to households in proportion to their holdings of corporate equity. The authors then compile the distribution of taxes paid, annually for the years 1960 to 2004, and consequently the distribution of post-tax income.

Post-tax income, so computed, does not correspond exactly to the theoretical concept that we used in sections 2 and 3 of *post-fisc* income. Thus, I have amended the Piketty-Saez data by including transfer payments, taken from the PSID, for the years 1974 – 2000. The statistics reported below are the *average post-fisc tax rates of US taxpayers*, by pre-fisc income quantile. This is defined as  $\frac{T - t}{y}$ , where  $T$  is the sum of the four taxes of Piketty-Saez,  $t$  is the value of transfer payments, and  $y$  is pre-fisc income. See the appendix for the details of how the Piketty-Saez data were amended.

In recent US fiscal history, the main tax reforms were the following<sup>12</sup>:

- In 1981, the Economic Reform Tax Act was passed under R. Reagan, which reduced the top marginal income tax rate from 70 to 50%, and continued to cut rates over three years;
- In 1986, the Tax Reform Act was signed by Reagan;
- In 1993, the top personal income tax rate was raised under B. Clinton to 39.6%;
- In 1997, the Tax Payer Relief Act cut the top rate on capital gains from 28 to 20%;
- In 2001, under George W. Bush, the Economic Growth and Tax Relief Reconciliation Act reduced the tax rate in the lowest bracket to 10%, reduced the highest marginal rate to 35%, and reduced the marriage penalty. In addition, the estate tax was to be reduced over a ten year period to the vanishing point.

Our model supposes that each of the two political parties proposes a tax policy as part of its electoral strategy. This is, of course, a stylization of reality. In order to confront the data, we will assume that when a major tax reform occurs, the policy that is enacted is the equilibrium policy of the president's party. That policy continues to hold until the next major tax reform. Thus, for example, we will assume that the policy in

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<sup>12</sup> See Brownlee (2004).

force prior to 1981 was the Democratic Party's equilibrium policy; that the policy after 1986 was the Republican Party's equilibrium policy; that the policy after 1993 was the Democratic Party's equilibrium policy; and that the policy after 2001 was the Republican Party's equilibrium policy. Our method will be to examine the *de facto* changes in the distribution of average post-fisc tax rates, before and after these major tax reforms, identifying the results that we observe with equilibrium party policies as described.<sup>13</sup>

Figure 7 (various panels) presents the average post-fisc tax rates before and after major tax reforms. In figure 7a, we present the average tax rates for the various quantile groups reported by Piketty and Saez, before (1981) and after (1988) the Reagan tax reforms. Piketty and Saez are particularly interested in the tax treatment of the very rich; we see that they disaggregate the top decile of the income distribution into six quantile groups, where the top group refers to the top 0.01% of the income distribution.

The main observations from figure 7a are that the Reagan tax reforms substantially reduced the tax rates on the top 0.5% of the income distribution, reduced tax rates on the top decile, left tax rates on the 60-90<sup>th</sup> quantile about the same, increased tax rates on the two quantiles occupying the 20- 60<sup>th</sup> percent, and reduced net taxation of the bottom quintile. If we interpret the pre- and post- Reagan reform tax rates as associated with Democratic and Republican equilibrium policies, respectively, these characteristics conform to the model's predictions, except for the treatment of the bottom quintile (due to the earned income tax credit which was expanded significantly in 1986): in particular, there is a sizeable group of middle income voters who receive essentially the same tax treatment by both parties. While *de jure* income-tax schedules are indeed piece-wise linear, we cannot assert that *de facto* conforms to a piece-wise linear rule.

One characteristic of our equilibrium policies that does not conform to the data is the predicted decrease in the average tax rate proposed by the Right party for the upper end of the distribution (incomes greater than  $h_2$  ; recall figure 4). In addition, the equilibrium policies in our model either tax at the minimal marginal tax rate (zero) or the maximal marginal tax rate ( $1 - \alpha$ ); this is not a feature of observed tax rates.

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<sup>13</sup> I am grateful to Kenneth Couch (University of Connecticut) and his research assistants, who extracted the transfer-payment data from the PSID for me.



Figure 7b presents the average post-fisc tax rates by quantile groups in 1988 and 1996, to attempt to capture the effect of the Clinton tax reform of 1993. Evidently, the Clinton reforms increased post-fisc tax rates on all quantiles. The quantiles occupying the 60-99<sup>th</sup> centiles had only small changes.

Figure 7c presents the tax rates before and after the G.W. Bush tax bill of 2001. The Bush tax bill appears to have reduced post-fisc tax rates for all quantiles except the bottom quintile, whose net tax rate has been increased. There appears to be no significant group whose tax treatment has stayed the same as it was in 1996. This is consistent with the view that G.W. Bush is not playing the game as it has been played in recent history, but is instead radically attempting to reduce the role of the federal government.

As a final contrast, we present in figure 7d tax rates in 1974 and 2004. These years are too far apart for us to interpret the tax treatments as a pair of equilibrium policies. The figure shows that there has been a large shift of the tax burden from the top 1% of the income distribution to the bottom 99%. However, pre-tax income has also shifted from the bottom 99% to the top 1% over this period, and so the shift in net taxes paid by the very rich and the rest will not have shifted so dramatically as the figure might suggest.

In figure 8, we present data on the percentage of voters who voted for the Democratic presidential candidate, for various election years, by income quantile<sup>14</sup>. In other words, the graphs in figure 8 give us a discrete approximation of the function  $\theta(\cdot; X^L, X^R)$  for various years. In 2000, 1996, and 1992, we can say that the swing voter types occupied a region between the 34<sup>th</sup> and 95<sup>th</sup> centiles of the income distribution; in 1980, swing voter types were between the 17<sup>th</sup> and 33<sup>rd</sup> centiles of the income distribution. It certainly appears from these data that the US is characterized by what we have called a left-right equilibrium: the function  $\theta$  is monotone decreasing. The general observation seems to be that over this period, the swing voter types have become more numerous, and have moved up in the income distribution. At least for the recent elections (since 1992), the regions of the income distribution where the two parties

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<sup>14</sup> I thank Joseph Bafumi for providing me with these data, which he compiled from the American National Election Studies (ANES).

coincide in their tax treatment (from Figure 7) seem to correspond roughly to where the swing voter types lie.

## 6. Discussion and conclusion

I have modeled political competition in a general election between two parties, incorporating two features of what appears to be American political reality: that parties compete on a very large policy space, and that their leaders appear to conflict internally over whether to represent their core voters, or to appeal to voter types sitting on the fence. In the application of the model to competition over tax policy, we chose the policy space to consist of all continuous functions restricted only by a budget constraint, and by a requirement that marginal tax rates lie everywhere in an interval  $[0, 1 - \alpha]$ . Choosing  $\alpha > 0$  was our simple strategy for capturing concerns with labor supply elasticity. In the simplest stationary equilibria of the model, the parties propose piece-wise linear post-fisc distributions of income, with the same treatment for what may be a quite large interval of middle-income voters. The more ‘swing-voter’ concerns dominate in the parties, the larger will this interval be. But even on the lower envelope of the equilibrium manifold, the policies will coincide for a non-negligible fraction of the income distribution.

We raise now the issue of multiple equilibria. A central problem in modeling political competition is conceiving of it in such a way that equilibria exist when policy spaces are multi-dimensional. We have solved that problem, but have instead a plethora of equilibria. Indeed, the bi-dimensionality of the equilibrium manifold here arises from there being, loosely speaking, two payoff functions in each party, associated with its two ‘factions.’ Roemer (1991, Chapter 8) shows that, in the related equilibrium concept of PUNE, the bi-dimensionality of the equilibrium manifold can be interpreted as due to there being two missing parameters in the model, which express the relative bargaining powers of the two factions in the two parties. (Every PUNE, in that model, can be generated as an outcome of a Nash-type bargaining game between factions, but with variable relative bargaining powers.) A similar interpretation is possible for the equilibria in this paper; I have not elaborated that here, as the move is similar to the one in the just-cited book, and because we lack data on these relative bargaining powers. If we had such data, we could test the predictions of the model more sharply.

We characterized *left-right* stationary equilibria. We remark that the characterization was fairly simple, mathematically speaking, because the proofs exploited heavily the monotonicity of the vote share function  $\theta$ . There may well be non-left-right stationary equilibria, in which the share function is non-monotonic: for instance, one party might win more than half the votes of the very poor and the very rich, while the other wins more than half of the middle income voters. These equilibria are much more difficult to study, and I did not attempt to do so here. In any case, figure 8 indicates that, in the US,  $\theta(\cdot)$  is indeed monotone decreasing.

Finally, we roughly tested the model's predictions by using the income-tax data assembled by Piketty and Saez (2006). Some of the features of our equilibria appear to hold, and some do not. Certainly legislated tax policies in the US are piece-wise linear; however, our model produces equilibrium policies with only two marginal tax rates, zero and  $1 - \alpha$ , and three pieces for each party (although different sets of pieces for the two parties). We described how we can generate equilibria with  $2n + 1$  linear pieces for any positive interger  $n$ . We found what might be termed mild support for the prediction that a middle income group receives the same tax treatment from both parties. It is clear that the Republican party taxes the rich less than does the Democratic party. It is not clear, however, that the Democrats always tax the bottom quintile more lightly (or transfer to them more heavily) than the Republicans. The tax treatment of this group is largely due to the impact of the earned income tax credit, which has been amended frequently. (It is also probably the case that labor-supply elasticity is of crucial importance in motivating legislators in their tax treatment of the poor, and our model almost ignores that issue.)

It is perhaps appealing to view certain aspects of tax policy as being due to *simplicity* or *inertia*: thus, one might conjecture that piece-wise linear policies with a small number of pieces are adopted for reasons of simplicity, and that policies do not change much between Left and Right administrations for a large group of middle-income voters for reasons of both simplicity (costly to change the entire tax code) and inertia. We have shown, however, that these characteristics of policies derive from political competition—they are equilibrium characteristics. We need not appeal to simplicity and inertia.

We also examined a model in which the swing-voter faction is replaced by a vote-share maximizing faction. We showed that the latter model possesses only one equilibrium, which is empirically unrealistic. We suggest that this adds credence to the main model of the paper.

One could point to many ways in which the model simplifies real politics. One of the most important is that actual tax policy is not proposed by parties in general elections: it is the consequence of legislation, and in particular, of legislative bargaining between the parties, and between the Congress and the executive branch. Another is that taxes and transfer payments are typically dealt with under separate pieces of legislation. Modeling the problem of tax policy as a legislative bargaining problem could improve the fit of reality to theory. Nevertheless, if we take Schumpeter's dictum seriously, as stated in the paper's epigram, and also Riker's (1982) dictum, that the most important moment of democracy is the general election, then the investigation reported upon here may shed some light on the problem.

### Appendix

#### Proof of Proposition 2:

1. Note that if  $X_a \in M_a(h_*)$  and  $X_b \in M_b(h_*)$  then

$$y = x_a + \alpha h_1 + h_* - h_1 \text{ and } y = x_b + h_*.$$

2. Write the budget constraint for a policy  $X \in M_a(h_*)$ :

$$x_a + \alpha \int_0^{h_1} h dF(h) + \alpha h_1 (1 - F(h_1)) + \int_{h_1}^{h_*} (h - h_1) dF(h) + (h_* - h_1)(1 - F(h_*)) + \alpha \int_{h_*}^{\infty} (h - h_*) dF(h) = \mu$$

We can rewrite this equation as:

$$x_a = (1 - \alpha) \left( \int_0^{h_1} h dF(h) + \int_{h_*}^{\infty} (h - h_*) dF(h) + h_1 (1 - F(h_1)) \right).$$

3. Viewing  $M_a(h_*)$  as parameterized by  $h_1$ , and differentiating the first expression for  $y$  in step 1 w.r.t.  $h_1$ , we have:

$$\frac{dy}{dh_1} = \frac{dx_a}{dh_1} - (1 - \alpha).$$

Now differentiating the expression derived in step 2 for  $x_a$  w.r.t.  $h_1$  gives:

$$\frac{dx_a}{dh_1} = (1 - \alpha)(1 - F(h_1)).$$

These two equations together tell us that:

$$\frac{dy}{dh_1} = (\alpha - 1)F(h_1) < 0.$$

Therefore the smallest (largest) value of  $y$  compatible with a policy's being in  $M_a(h_*)$  is associated with  $h_1 = h_*$  (respectively,  $h_1 = 0$ ). Using the equation for  $x_a$  in step 2, we have:

$$x_a(h_*) = (1 - \alpha)\mu, \quad x_a(0) = (1 - \alpha) \int_{h_*}^{\infty} (h - h_*) dF(h),$$

and so these two values of  $y$  are given by:

$$y_a(h_*) = (1 - \alpha)\mu + \alpha h_*, \quad y_a(0) = h_* + (1 - \alpha) \int_{h_*}^{\infty} (h - h_*) dF(h).$$

4. We perform a similar analysis of policies in  $M_b(h_*)$ . For any such policy, we may rewrite the budget constraint as:

$$x_b = (1 - \alpha) \int_{h_*}^{h_2} h dF(h) + (1 - \alpha)h_2(1 - F(h_2)) - (1 - \alpha)h_*(1 - F(h_*)).$$

Differentiating this equation w.r.t. the parameter  $h_2$  gives:

$$\frac{dx_b}{dh_2} = (1 - \alpha)(1 - F(h_2)) > 0;$$

now using the expression for  $y$  in step 1, we have:

$$\frac{dy}{dh_2} = \frac{dx_b}{dh_2} > 0.$$

Therefore, the smallest (largest) value of  $y$  compatible with a policy's being in  $M_b(h_*)$  is associated with  $h_2 = h_*$  (respectively,  $h_2 = \infty$ ). These two values of  $y$  are:

$$y_b(h_*) = h_*, \quad y_b(\infty) = (1 - \alpha) \int_{h_*}^{\infty} (h - h_*) dF(h) + h_*.$$

5. To summarize, the number  $y$  is associated with a policy in  $M_a(h_*)$  if and only if

$$y_a(h_*) \leq y \leq y_a(0),$$

and  $y$  is associated with a policy in  $M_b(h_*)$  if and only if

$$y_b(h_*) \leq y \leq y_b(\infty).$$

Notice that  $y_a(0) = y_b(\infty)$ ; parts A and B of the proposition follow immediately.

6. We prove part C. We have shown that the smallest value of  $x_a$  is

$$(1 - \alpha) \int_{h_*}^{\infty} (h - h_*) dF(h) \text{ which is positive, as long as } F \text{ has some support on } (h_*, \infty). \text{ The}$$

argument in step 4 above shows that  $x_b > 0$  except in the singular case that  $h_* = h_2$ . In

that case, the policy  $X_b$  is the laissez-faire policy  $X_b(h) = h$ . ■

Proof of Theorem 1:

1. The theorem will be proved if we can show that  $X^a$  and  $X^b$  solve the programs in conditions (β2) and (β3) of definition 2, respectively. We first address (β2). Let the density function of  $F$  be denoted  $f$ . The numbers  $h_1$  and  $h_2$  come with the functions  $X^a$  and  $X^b$ .

2. Define the number  $\rho$ , the functions  $r(h)$  on  $[0, h_1]$ ,  $s(h)$  on  $[h_1, h_*]$  and  $t(h)$  on  $[h_*, \infty)$ , and the number  $\lambda$  as follows.

$$(i) \quad \rho = \frac{\int_0^{h_1} \theta(h) dF(h)}{F(h_1)},$$

$$(ii) \quad r(0) = 0 \text{ and } r'(h) = (\theta(h) - \rho)f(h) \text{ on } [0, h_1],$$

$$(iii) \quad s(h_1) = 0 \text{ and } s'(h) = (\rho - \theta(h))f(h) \text{ on } [h_1, h_*],$$

$$(iv) \quad t(h_*) = \rho(1 - F(h_*)) - \int_{h_*}^{\infty} \theta(h) dF(h) \text{ and}$$

$$t'(h) = (\theta(h) - \rho)f(h) \text{ on } [h_*, \infty)$$

$$(v) \quad \lambda = s(h_*) + t(h_*).$$

Note that  $\rho > 0$ . Note, from Proposition 3, that the function  $r$  is first increasing

and then decreasing. Compute that  $r(h_1) = r(0) + \int_0^{h_1} r'(h)dh = 0$ . Therefore  $r$  is a non-

negative function on its domain. Note from Proposition 3 that  $s$  is an increasing function on its domain: since  $\theta$  is constant on  $[h_1, h_*]$ , by Proposition 3, we know that  $\rho > \theta(h)$  on this interval. Therefore  $s$  is a non-negative function on its domain, and

$s(h_*) = \rho(F(h_*) - F(h_1)) - \int_{h_1}^{h_*} \theta(h)dF(h) > 0$ . Note that  $t$  is decreasing on its domain, and

$t(\infty) = t(h_*) + \int_{h_*}^{\infty} t'(h)dh = 0$ . Therefore  $t$  is non-negative on its domain. Finally, note that

$\lambda > 0$ .

4. Suppose that  $X^a$  were not the solution to the program ( $\beta 2$ ) of definition 2, and that the true solution is some other policy  $X$ . Define the function  $g$  by the equation

$X(h) = X^a(h) + g(h)$ . Now define the function  $\Delta : \mathbb{R} \rightarrow \mathbb{R}$  as follows.

$$\begin{aligned} \Delta(\varepsilon) = & \int_0^{\infty} (X^a(h) + \varepsilon g(h))\theta(h)dF(h) + \int_0^{h_1} (X^{a'}(h) + \varepsilon g'(h) - \alpha)r(h)dh + \\ & \int_{h_1}^{h_*} (1 - (X^{a'}(h) + \varepsilon g'(h)))s(h)dh + \int_{h_*}^{\infty} (X^{a'}(h) + \varepsilon g'(h) - \alpha)t(h)dh + \\ & \lambda(X^a(h_*) + \varepsilon g(h_*) - y) + \rho\left(\mu - \int_0^{\infty} (X^a(h) + \varepsilon g(h))dF(h)\right) \end{aligned}$$

Note that  $\Delta$  is a linear function, and that  $\Delta(0) = \int_0^{\infty} X^a(h)\theta(h)dF(h)$ : this is the objective

of program ( $\beta 2$ ) evaluated at the policy  $X^a$ . Note as well that when  $\varepsilon = 1$ , all the terms

in the expression defining  $\Delta$  are non-negative: this follows from the fact that

$r, s, t, \lambda$  and  $\rho$  are all non-negative functions or numbers, and that  $X \in \mathfrak{S}$ . Suppose we

can show that  $\Delta'(0) = 0$ : then  $\Delta$  will be equal to a constant, and consequently

$\Delta(0) = \Delta(1)$ . But this implies that the value of the objective function of ( $\beta 2$ ) at  $X^a$  is at

least as large as its value at  $X$ : a contradiction. Thus we will have proved that  $X^a$  solves program ( $\beta 2$ ) if we can show that  $\Delta'(0) = 0$ .

5. Compute that

$$\begin{aligned} \Delta'(0) &= \int_0^\infty \theta(h)g(h)dF(h) + \int_0^{h_1} g'(h)r(h)dh - \int_{h_1}^{h_*} g'(h)s(h)dh \\ &+ \int_{h_*}^\infty g'(h)t(h)dh + \lambda g(h_*) - \rho \int_0^\infty g(h)dF(h) \end{aligned}$$

Hence, integrating three times by parts, we have:

$$\begin{aligned} \Delta'(0) &= \int_0^\infty \theta(h)g(h)dF(h) + g(h)r(h)\Big|_0^{h_1} - \int_0^{h_1} r'(h)g(h)dh - g(h)s(h)\Big|_{h_1}^{h_*} \\ &+ \int_{h_1}^{h_*} s'(h)g(h)dh + g(h)t(h)\Big|_{h_*}^\infty - \int_{h_*}^\infty t'(h)g(h)dh + \lambda g(h_*) - \rho \int_0^\infty g(h)dF(h). \end{aligned}$$

We next re-group terms and write:

$$\begin{aligned} \Delta'(0) &= \int_0^{h_1} ((\theta(h) - \rho)f(h) - r'(h))g(h)dh + \int_{h_1}^{h_*} (s'(h) - (\rho - \theta(h))f(h))g(h)dh + \\ &\int_{h_*}^\infty ((\theta(h) - \rho)f(h) - t'(h))g(h)dh + g(h_*)(\lambda - s(h_*) - t(h_*)) - g(0)r(0) + g(h_1)(r(h_1) + s(h_1)) \\ &+ t(\infty)g(\infty). \end{aligned}$$

Now check, by the definitions of  $r, s, t$  and  $\lambda$  that every term on the r.h.s. of this equation vanishes, which proves that  $\Delta'(0) = 0$ .

6. We proceed to prove that  $X^b$  is the solution to program ( $\beta 3$ ) of definition 2.

Suppose that the true solution is  $X$  and now define the function  $g$  by  $X = X^b + g$ . We now define functions  $R, S$ , and  $T$ , and numbers  $\gamma$  and  $\delta$  as follows:

$$(i) \quad \delta = \int_{h_2}^\infty (1 - \theta(h))dF(h) / (1 - F(h_2)),$$

$$(ii) \quad R(0) = 0 \text{ and } R'(h) = (\delta - (1 - \theta(h)))f(h) \text{ on } [0, h_*],$$



$$(iii) S(h_*) = \int_{h_2}^{h_*} (\delta - (1 - \theta(h)))dF(h) \text{ and } S'(h) = (1 - \theta(h) - \delta)f(h) \text{ on } (h_*, h_2),$$

$$(iv) T(h_2) = 0 \text{ and } T'(h) = (\delta - (1 - \theta(h)))f(h) \text{ on } (h_2, \infty),$$

$$(v) \gamma = R(h_*) + S(h_*).$$

Since the function  $1 - \theta(h)$  is (weakly) increasing (see Proposition 3), it follows from the definition of  $\delta$  that  $R' \geq 0, S' \leq 0$ , and that  $S'(h_2) = 0$ . The functions  $R, S$ , and  $T$  are non-negative on their domains. As well,  $R(h_*), S(h_*)$  and  $\gamma$  are positive.

7. We now define the function  $\Phi$  by:

$$\begin{aligned} \Phi(\varepsilon) = & \int_0^{\infty} (1 - \theta(h))(X^b(h) + \varepsilon g(h))dF(h) + \int_0^{h_*} \left(1 - (X^{b'}(h) + \varepsilon g'(h))\right)R(h)dh + \\ & \int_{h_*}^{h_2} \left(X^{b'}(h) + \varepsilon g'(h) - \alpha\right)S(h)dh + \int_{h_2}^{\infty} \left(1 - (X^{b'}(h) + \varepsilon g'(h))\right)T(h)dh + \gamma(X^b(h_*) + \varepsilon g(h_*)) \\ & + \delta \left( \mu - \int_0^{\infty} (X^b(h) + \varepsilon g(h))dF(h) \right). \end{aligned}$$

All the terms on the r.h.s. of this equation are non-negative, and so, as we argued above, if we can demonstrate that  $\Phi'(0) = 0$ , then we will have proved that  $X^b$  solves the program in condition ( $\beta_3$ ) of definition 2.

8. Compute that

$$\begin{aligned} \Phi'(0) = & \int_0^{\infty} (1 - \theta(h))g(h)dF(h) - g(h)R(h)\Big|_0^{h_*} + \int_0^{h_*} R'(h)g(h)dh + g(h)S(h)\Big|_{h_*}^{h_2} \\ & - \int_{h_*}^{h_2} S'(h)g(h) - g(h)T(h)\Big|_{h_*}^{\infty} + \int_{h_2}^{\infty} T'(h)g(h)dh + \gamma g(h_*) - \delta \int_0^{\infty} g(h)dF(h). \end{aligned}$$

Re-grouping terms, we have:

$$\begin{aligned} \Phi'(0) = & \int_0^{h_*} ((1 - \theta(h) - \delta)f(h) + R'(h))g(h)dh + \int_{h_*}^{h_2} ((1 - \theta(h) - \delta)f(h) - S'(h))g(h)dh + \\ & \int_{h_2}^{\infty} ((1 - \theta(h) - \delta)f(h) + T'(h))g(h)dh + (\gamma - R(h_*) - S(h_*)) - g(0)R(0) + g(h_2)(S(h_2) + T(h_2)) \\ & - g(\infty)T(\infty). \end{aligned}$$

From the definitions of the functions  $R, S, T$  and the numbers  $\delta$  and  $\gamma$ , we observe that all terms on the r.h.s. of this equation vanish, which proves the theorem<sup>15</sup>. ■

### Proof of Theorem 2:

1. It is clear that the ideal policy for a type  $h_*$  -- the policy in  $\mathfrak{S}$  that maximizes its (post-fisc) income -- has some value  $y$  at  $h_*$ , increases as slowly as possible for  $h > h_*$ , and decreases from the  $(h_*, y)$  as rapidly as possible for  $h < h_*$ . This is the way to spend as few resources as possible on everyone other than  $h_*$ . Thus the ideal policy for  $h_*$  is defined by:

$$X^*(h) = \begin{cases} x_0 + h, & h \leq h_* \\ x_0 + h_* + \alpha(h - h_*), & h > h_* \end{cases}$$

where  $x_0$  is such that this policy integrates to  $\mu$ . But this is precisely the policy in  $M_a(h_*) \cap M_b(h_*)$  when  $y = y_{\max}(h_*)$ .

2. If  $h_* \leq \mu$  and  $y = (1 - \alpha)\mu + \alpha h_*$  then the policy  $X^a \in M_a(h_*)$  is a line of slope  $\alpha$  such that  $x_a = (1 - \alpha)\mu$ . We prove below (see the proof of theorem 5), using the variational technique of the proof of theorem 1, that this is the policy that maximizes

$\int \theta(h)X(h)dF(h)$  on  $\mathfrak{S}$ . Moreover, the fact is intuitively clear. Because  $\theta$  is a

decreasing function and the objective functional is linear in  $X$ , the objective wishes to push resources as much as possible to the poorest. The solution is to maximize what is given to  $h=0$ , which means to increase as slowly as possible (that is, at rate  $\alpha$ ) on the

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<sup>15</sup> The convention that “ $t(\infty) = 0 = T(\infty)$ ” is short-hand for a transversality condition.

The proof can be rigorously completed by checking that  $\lim_{h \rightarrow \infty} t(h)g(h) = 0$  and

$\lim_{h \rightarrow \infty} T(h)g(h) = 0$ : these claims are true.

whole positive line, subject to having given just enough to  $h=0$  so that the policy integrates to  $\mu$ .

3. If  $h_* \geq \mu$  and  $y = y_{\min}(h_*) = h_*$  then the policy  $X^b \in M_b(h_*)$  is the laissez-faire policy  $X(h) = h$ . It is also intuitively clear that this is the policy that maximizes

$\int (1 - \theta(h))X(h)dF(h)$ : for now, the objective wishes to push resources to the very rich.

Once it is decided how much the very rich get, the strategy must be to decrease as fast as possible (i.e., at rate one) for  $h$  smaller. This yields in the limit the laissez-faire policy.

Of course, this can also be proved using the variational technique of theorem 1. ■

### Proof of Theorem 3:

1. In the political contest at date 1, the L party (the one whose constituency is defined by the function  $\theta_0$ ) solves this program:

$$\begin{aligned} & \max_{X \in \mathfrak{S}} \int \theta_0(h)X(h)dF(h) \\ & \text{s.t.} \\ & X(h_*) \geq X^R(h_*) \end{aligned}$$

since the unique swing type at date 0 is  $h_*$ . There is a similar program for the R party.

Thus the political equilibrium at date one is exactly a triple  $(X^L, X^R, y)$  such that:

- (1)  $X^L$  maximizes  $\int \theta_0(h)X(h)dF(h)$  subject to  $X(h_*) \geq y$ ;
- (2)  $X^R$  maximizes  $\int (1 - \theta_0(h))X(h)dF(h)$  subject to  $X(h_*) \geq y$ ;
- (3)  $X^L(h_*) = y = X^R(h_*)$ .

This looks almost like a 1-stationary equilibrium, except that the function  $\theta_0$  is not related to the vote shares engendered by the policies  $X^L$  and  $X^R$ . Now suppose we choose  $y \in [y_{\min}(h_*), y_{\max}(h_*)]$  so that  $(h_*, y) \in \Gamma$ . Note, by examining the proof of theorem 1, that the only fact invoked about the function  $\theta_*$  was that it was weakly monotone decreasing. So the argument of theorem will apply just as well if we substitute the decreasing function  $\theta_0$  for  $\theta_*$ . Hence, by the same argument as in theorem 1, the

functions  $X^L$  and  $X^R$  which satisfy conditions (1)-(3) above are an ordered pair in the set  $M_a(h_*) \times M_b(h_*)$ . This proves the first statement in the theorem.

2. Conversely, let  $(X^L, X^R)$  be any equilibrium (stationary or not) at date 1 emanating from  $\theta_0$ . The policies have a common value  $y = X^L(h_*) = X^R(h_*)$  since  $h_*$  was the unique swing voter. If  $(h_*, y) \in \Gamma$ , then again the optimization of each party yields inexorably to the ordered pair of policies  $M_a(h_*) \times M_b(h_*)$  associated with the income  $y$  at the pivot  $h_*$ .

Thus, what must be shown is that  $y = X^L(h_*) \in [y_{\min}(h_*), y_{\max}(h_*)]$ . Define the following two functions:

$$X^{\max}(h) = \begin{cases} (y - \alpha h_*) + \alpha h, & 0 \leq h \leq h_* \\ y - h_* + h, & h > h_* \end{cases}$$

$$X^{\min}(h) = \begin{cases} y - h_* + h, & 0 \leq h \leq h_* \\ y + \alpha(h - h_*), & h > h_* \end{cases}$$

Subject to passing through the point  $(h_*, y)$  and satisfying conditions (P0) and (P1) of the definition of policies in  $\mathfrak{S}$ ,  $X^{\max}$  is the way of allocating income which consumes the maximal amount of resource and  $X^{\min}$  is the way of allocating income which consumes the minimal amount of resource. It therefore follows that

$$\int X^{\max}(h)dF(h) \geq \mu \text{ and } \int X^{\min}(h)dF(h) \leq \mu.$$

But this is precisely the condition for  $(h_*, y)$  to be in  $\Gamma$ . ■

#### Proof of Theorem 4:

1. We prove the converse part first. Let  $(X^L, X^R)$  be the date-1 equilibrium in a sequence of historical equilibrium beginning with the datum  $\theta_0$ . Define for  $h \in [h_*, h_{**}]$ ,  $Z(h) \equiv X^L(h)$ . From the definition of equilibrium, it follows that for  $h \in [h_*, h_{**}]$ ,  $X^R(h) = Z(h)$  as well. Suppose that the functions  $(\hat{X}^L, \hat{X}^R)$  defined in the theorem's statement can be defined and integrate to  $\mu$ . We show that  $X^L$  and  $X^R$  are precisely the functions  $\hat{X}^L$  and  $\hat{X}^R$ . Suppose, to the contrary, that  $X^L \neq \hat{X}^L$ . Then define the (non-zero) function  $g$  by  $X^L(h) = \hat{X}^L(h) + g(h)$ .

Let the functions  $r(\cdot), s(\cdot), t(\cdot)$ , and  $u(\cdot)$  be some non-negative functions, defined on the intervals given by the limits of the integrals in which they appear below, and let  $\lambda_1, \lambda_2$  and  $\rho$  be arbitrary non-negative numbers. Now define the function:

$$\begin{aligned} \Delta(\varepsilon) = & \int_0^{\infty} \theta_0(h)(\hat{X}^L(h) + \varepsilon g(h))dF(h) + \int_0^{h_1} \varepsilon g'(h)r(h)dh - \int_{h_1}^{h_*} \varepsilon g'(h)s(h)dh \\ & + \lambda_1 \varepsilon g(h_*) + \int_{h_*}^{h_{**}} \varepsilon g(h)t(h)dh + \lambda_2 \varepsilon g(h_{**}) + \int_{h_{**}}^{\infty} \varepsilon g'(h)u(h)dh - \rho \int_0^{\infty} \varepsilon g(h)dF(h). \end{aligned}$$

Note that  $\Delta(0)$  is the value of the L party's objective at date 1 evaluated at  $\hat{X}^L$  and  $\Delta(1)$  is the value of the L party's objective at  $X^L$  plus a series of terms all of which must be non-negative. (E.g., in the interval  $[0, h_1]$ ,  $g' \geq 0$  because  $(\hat{X}^L)'(h) \equiv \alpha$  on this interval, and so on for all the other terms.)

Suppose we can choose  $r, s, t, u, \lambda_1, \lambda_2$ , and  $\rho$  so that  $\Delta'(0) = 0$ . Since  $\Delta$  is a linear function, it will follow that it is maximized at  $\varepsilon = 0$ ; in particular,  $\Delta(0) \geq \Delta(1)$ . This implies, *a fortiori*, that the value of the L party's objective is at least as great at  $\hat{X}^L$  as at  $X^L$ , which will be the desired contradiction.

2. Calculate, using integration by parts, that:

$$\begin{aligned} \Delta'(0) = & \int_0^{\infty} \theta_0(h)g(h)dF(h) + g(h)r(h)\Big|_0^{h_1} - \int_0^{h_1} g(h)r'(h)dh - g(h)s(h)\Big|_{h_1}^{h_*} + \\ & \int_{h_1}^{h_*} g(h)s'(h)dh + \lambda_1 g(h_*) + \int_{h_*}^{h_{**}} g(h)t(h)dh + \lambda_2 g(h_{**}) + g(h)u(h)\Big|_{h_{**}}^{\infty} - \int_{h_{**}}^{\infty} g(h)u'(h)dh \\ & - \rho \int_0^{\infty} g(h)dF(h); \end{aligned}$$

now organize the terms above to express:

$$\begin{aligned} \Delta'(0) = & \int_0^{h_1} [\theta_0(h)f(h) - \rho f(h) - r'(h)]g(h)dh + \int_{h_1}^{h_*} [\theta_0(h)f(h) - \rho f(h) + s'(h)]g(h)dh + \\ & \int_{h_*}^{h_{**}} [\theta_0(h)f(h) - \rho f(h) + t(h)]g(h)dh + \int_{h_{**}}^{\infty} [\theta_0(h)f(h) - \rho f(h) - u'(h)]g(h)dh + r(0)g(0) + \\ & g(h_1)(r(h_1) + s(h_1)) + g(h_*)(\lambda_1 - s(h_*)) + g(h_{**})(\lambda_2 - u(h_{**})) + g(\infty)u(\infty), \end{aligned}$$

where  $f(h) \equiv dF(h)$ .

Therefore, we can annihilate all these terms if the ‘Lagrangian functions and multipliers’ are chosen to fulfill the following equations:

- (a)  $r'(h) = (\theta_0(h) - \rho)f(h)$  on  $[0, h_1]$
- (b)  $s'(h) = (\rho - \theta_0(h))f(h)$  on  $[h_1, h_*]$
- (c)  $t(h) = (\rho - \theta_0(h))f(h)$  on  $[h_*, h_{**}]$
- (d)  $u'(h) = (\theta_0(h) - \rho)f(h)$  on  $[h_*, h_{**}]$
- (e)  $r(0) = 0 = r(h_1) = s(h_1)$
- (f)  $\lambda_1 = s(h_*)$
- (g)  $\lambda_2 = u(h_{**})$
- (h)  $u(\infty) = 0$ .

3. Since  $r$  must be zero at its endpoints (statement (e)), using (a), we define:

$$\rho = \theta^{av}[0, h_1] \equiv \frac{\int_0^{h_1} \theta_0(h) dF(h)}{F(h_1)}.$$

Since  $\theta_0$  is a weakly decreasing function, it follows that  $r$  is non-negative on  $[0, h_1]$ .

Obviously  $\rho > 0$ . It now follows from statement (b) that  $s' \geq 0$  on  $[h_1, h_*]$ , again

invoking the fact that  $\theta_0$  is decreasing. Hence,  $s(h_*) = \int_0^{h_1} s'(h) dh \geq 0$  (here we use the

fact that we choose  $s(h_1) = 0$ ). Hence from (f),  $\lambda_1 \geq 0$  and  $s$  is a non-negative function

on its domain. From (c),  $t(h)$  is a non-negative function, again invoking the fact that  $\theta_0$

is decreasing. Now from (d),  $u$  must be a decreasing function on  $[h_{**}, \infty)$  and must

converge to zero at infinity, so we define:

$$u(h_{**}) = \int_{h_{**}}^{\infty} (\rho - \theta_0(h)) dh > 0.$$

Hence from (g),  $\lambda_2 > 0$ .

Hence the Lagrangian functions and multipliers have been defined, to be non-negative, and to fulfill the conditions (a) – (h), proving the claim<sup>16</sup>.

A similar argument shows that  $X^R = \hat{X}^R$ .

4. Finally, we remark that indeed the functions  $(\hat{X}^L, \hat{X}^R)$  can be defined and integrate to  $\mu$ : this argument is just like the one presented in the proof of theorem 3.
5. We have shown that any date-1 equilibrium with the historical vote-share function  $\theta_0$  is of the form  $(\hat{X}^L, \hat{X}^R)$ . The first statement in the theorem is clearly proved by the same technique. That is, if the functions  $(\hat{X}^L, \hat{X}^R)$  can be defined (which is true if the integral of  $Z$  on its domain is not too small or too large) then they comprise a stationary equilibrium reached at date 1. To show stationarity, we need only observe that the vote share function  $\hat{\theta}(\cdot)$  defined by  $(\hat{X}^L, \hat{X}^R)$  is itself monotone decreasing, and the same optimization proof works. ■

#### Proof of theorem 5:

##### Part A.

1. We show that  $\hat{X}$  maximizes vote share when competing against  $\hat{X}$ . Suppose this were false. Then there exists a policy  $X = (\hat{X} + g) \in \mathfrak{S}$  such that

$$\int S(\hat{X}(h) + g(h), \hat{X}(h)) dF(h) > 0.5 .$$

To show this is impossible, we define the function:

$$\begin{aligned} \Delta(\varepsilon) = & \int_0^\infty S(\hat{X}(h) + \varepsilon g(h), \hat{X}(h)) dF(h) + \rho(\mu - \int (\hat{X}(h) + \varepsilon g(h)) dF(h)) + \\ & \int (\hat{X}'(h) + \varepsilon g'(h) - \alpha)r(h) dh. \end{aligned}$$

By premise (S5),  $\Delta$  is concave. It follows that if we can choose  $\rho \geq 0$  and  $r \geq 0$  such that  $\Delta'(0) = 0$ , then  $\Delta$  is maximized at  $\varepsilon = 0$ , a contradiction.

Compute, using integration by parts, that:

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<sup>16</sup> As we remarked in the proof of theorem 1, the statement " $u(\infty) = 0$ " is short-hand for the statement  $\lim_{h \rightarrow \infty} g(h)u(h) = 0$ , which is true.

$$\Delta'(0) = \int S_1(\hat{X}(h), \hat{X}(h))g(h)dF(h) - \rho \int g(h)dF(h) + g(h)r(h)\Big|_0^\infty \\ - \int r'(h)g(h)dh = \int \left[ (S_1(\hat{X}(h), \hat{X}(h)) - \rho)f(h) - r'(h) \right] g(h)dh + gr\Big|_0^\infty.$$

Hence it is only necessary to choose  $r(\cdot)$  and  $\rho$  so that:

$$r'(h) = (S_1(\hat{X}(h), \hat{X}(h)) - \rho)f(h) \text{ and } r(0) = 0 = r(\infty).$$

Choose  $r(0) = 0$  and define  $\rho = \int S_1(\hat{X}(h), \hat{X}(h))dF(h)$ ; then  $r(\infty) = 0$  by integration.

Moreover, since  $S_1(\hat{X}(h), \hat{X}(h))$  is a (weakly) decreasing function of  $h$  (see postulate (S4)), it follows that  $r'$  is initially non-negative and finally non-positive, so that, because of the end-point conditions,  $r$  is a non-negative function. This shows that  $\hat{X}$  maximizes vote share against itself.

2. Furthermore,  $\hat{X}$  is the unique vote-share maximizer against  $\hat{X}$ , since  $S(x, y)$  is concave but not linear in  $x$ . Hence any other policy running against  $\hat{X}$  receives less than half the vote, proving that  $\hat{X}$  is a strict Condorcet winner.

Part B. Here we define the function:

$$\Delta(\varepsilon) = \int_0^\infty \theta(h)(\hat{X}(h) + \varepsilon g(h))dF(h) + \rho \left( \mu - \int (\hat{X}(h) + \varepsilon g(h))dF(h) \right) + \\ \int_0^\infty \left( \hat{X}'(h) + \varepsilon g'(h) - \alpha \right) r(h)dh.$$

If we can choose  $r$  and  $\rho$  non-negative such that  $\Delta'(0) = 0$ , then part B is shown.

Compute that  $\Delta'(0) = \int_0^\infty [(\theta(h) - \rho)f(h) - r'(h)]g(h)dh + gr\Big|_0^\infty$ . Define

$r'(h) = (\theta(h) - \rho)f(h)$  and  $r(0) = 0$  and  $\rho = \int \theta(h)dF(h)$ . Then  $r(\infty) = 0$  by integration.

Since  $\theta$  is weakly decreasing, it follows that  $r \geq 0$ , and part B is proved.

Part C.

1. Suppose that  $(\theta, X^L, X^R)$  is a Left-Right alternative equilibrium and  $\theta$  is not identically equal to one-half (i.e.,  $X^L \neq X^R$ ). Since both parties must win one-half the vote, we have  $X^L \neq \hat{X} \neq X^R$ , since  $\hat{X}$  is a strict Condorcet winner. By Parts B and A,



$\hat{X}$  maximizes  $\int \theta(h)X(h)dF(h)$ , and achieves more than half the vote against  $X^R$ .

Hence the Left party should deviate to  $\hat{X}$ ; hence, this is not an equilibrium.

2. Suppose that  $(\frac{1}{2}, X, X)$  is an alternative equilibrium and  $X \neq \hat{X}$ . Then either party should deviate to  $\hat{X}$ : the average income of its constituency will remain unchanged at  $\mu$ , but vote share will increase.

3. Finally, it is obvious that  $(\frac{1}{2}, \hat{X}, \hat{X})$  is an alternative equilibrium. ■

### Amending the Piketty – Saez (2006) data to include transfer payments

I explain how we amended the Piketty-Saez data to attain the average post-fisc tax rate for a quantile – here, for the bottom quintile. Let:

$x_{20}$  = post-tax income of bottom quintile

$y_{20}$  = pre-tax income of bottom quintile

$\bar{x}$  = average post-tax income of whole sample

$\bar{y}$  = average pre-tax income of whole sample

$T_{20}$  = average taxes paid by bottom quintile

$t_{20}$  = average transfers received by bottom quintile

$\bar{T}$  = average taxes paid, whole sample

$\bar{t}$  = average transfers received, whole sample

We wish to compute

$$q_{20} = \text{average post-fisc tax rate of bottom quintile} = \frac{T_{20} - t_{20}}{y_{20}}.$$

Piketty-Saez (2006) give us the pre-tax income share

$$r_{20} = \frac{.20y_{20}}{\bar{y}} \quad (1),$$

and the average tax rate for the bottom quintile

$$u_{20} = \frac{T_{20}}{y_{20}} \quad (2).$$

Table A0 of Piketty-Saez gives the average nominal income, which is  $\bar{y}$ .

Hence, from (1), we can compute  $y_{20}$ ; from (3) we can compute  $T_{20}$ . We extract  $t_{20}$  from the PSID, and thus compute  $q_{20}$ .

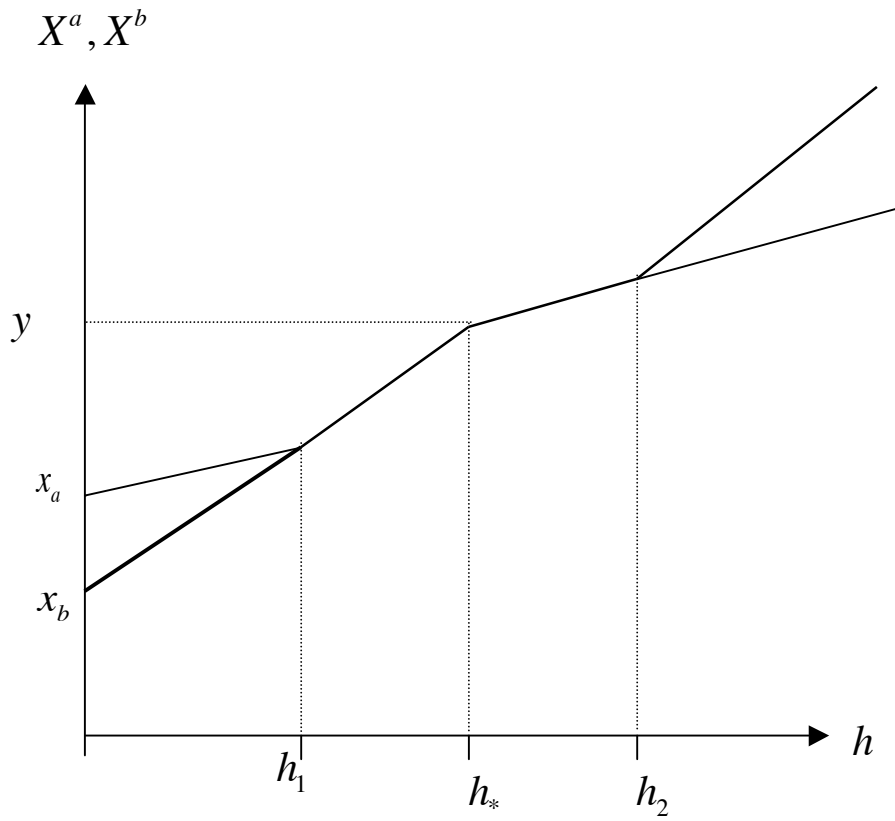
For 1974, Piketty-Saez (2006) does not provide the necessary data for the quantiles in the bottom 90% of the income distribution. We proceeded as follows. First, we calculated the income shares from the Current Population Survey in 2001 for the quantiles in the bottom 90%, and also those shares from the same source in 2004. Then we applied the factors by which those shares changed to the income shares provided by Piketty-Saez (2006) for 2001. This gave us the necessary income shares for all quantiles in 2004, which permitted us to compute the values shown in figure 7c.

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Figure 1 Policies  $X^a \in M_a(h_*)$  [thin line] and  $X^b \in M^b(h_*)$  [bold line] which share a common value  $y$  at  $h_*$



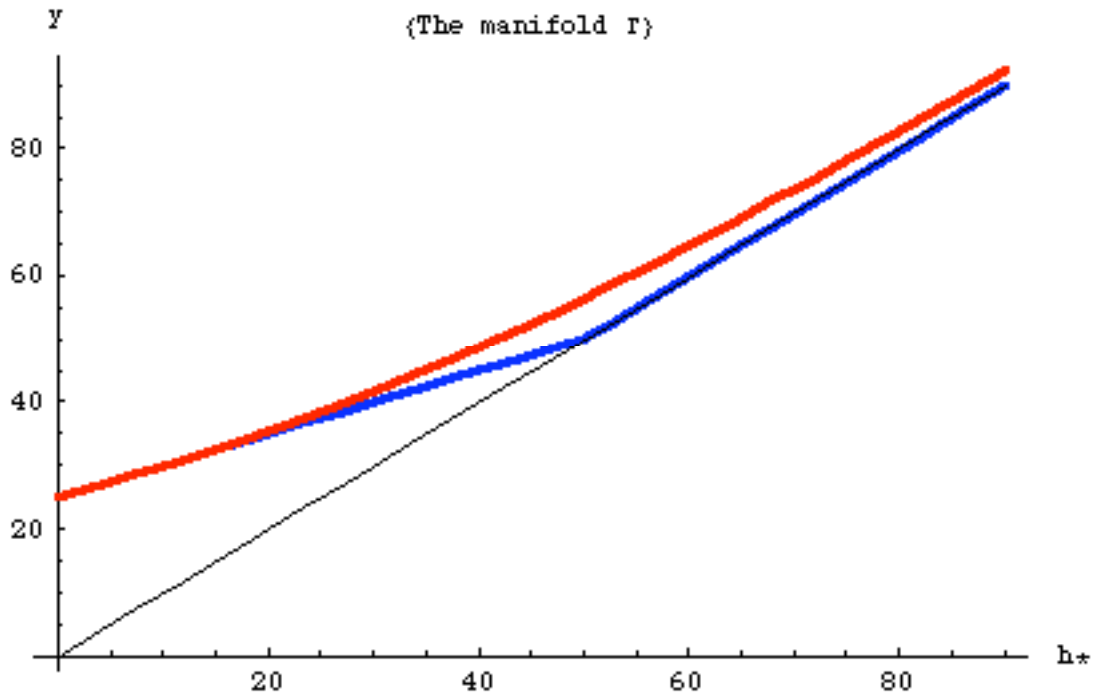


Figure 2 Graphs of the functions  $y^{\max}(\cdot), y^{\min}(\cdot)$  and the ray  $y = h_*$ . The manifold  $\Gamma$  is the set bounded by the two bold curves.

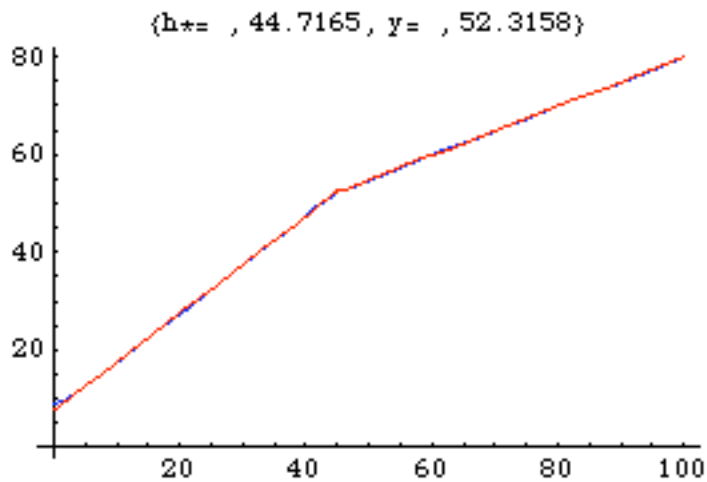


Figure 3a  $L$  and  $R$  policies on the upper envelope of the manifold  $\Gamma$

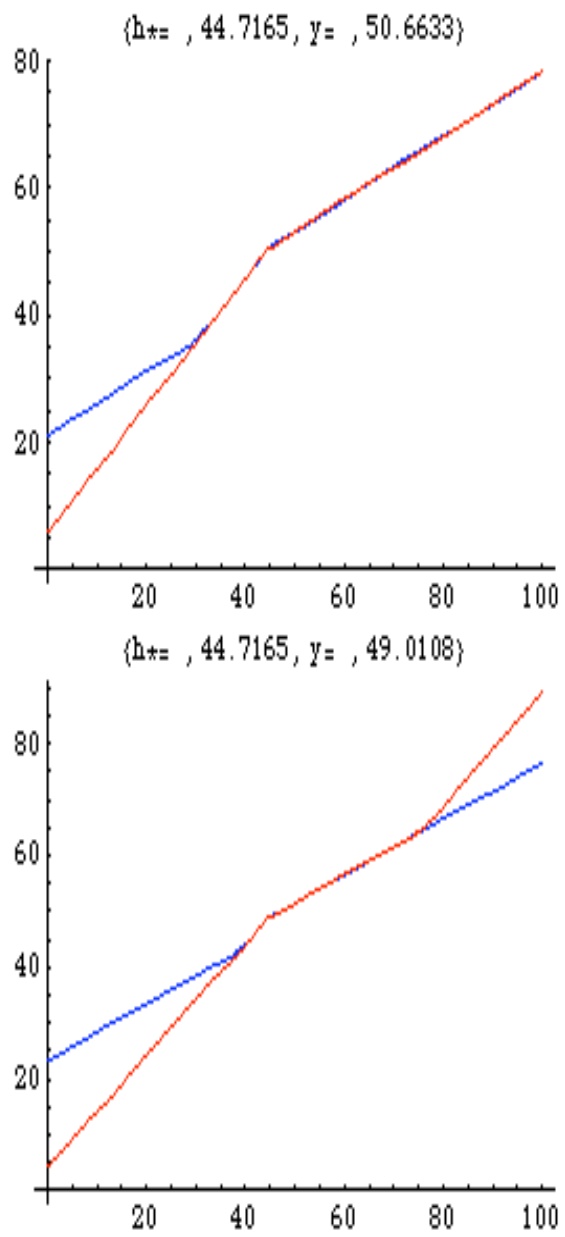


Figure 3b. Graphs of stationary equilibria as  $y$  decreases in  $\Gamma$  at constant  $h_*$ .

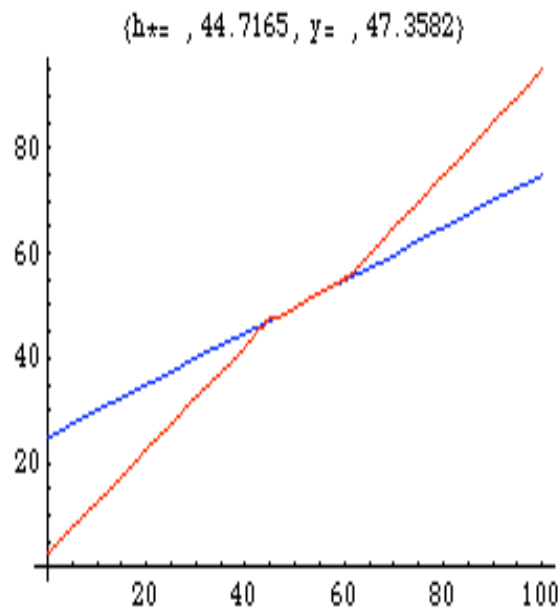


Figure 3c. Stationary equilibrium on the lower envelope of the  $\Gamma$ , where Left plays the ideal policy of its average constituency

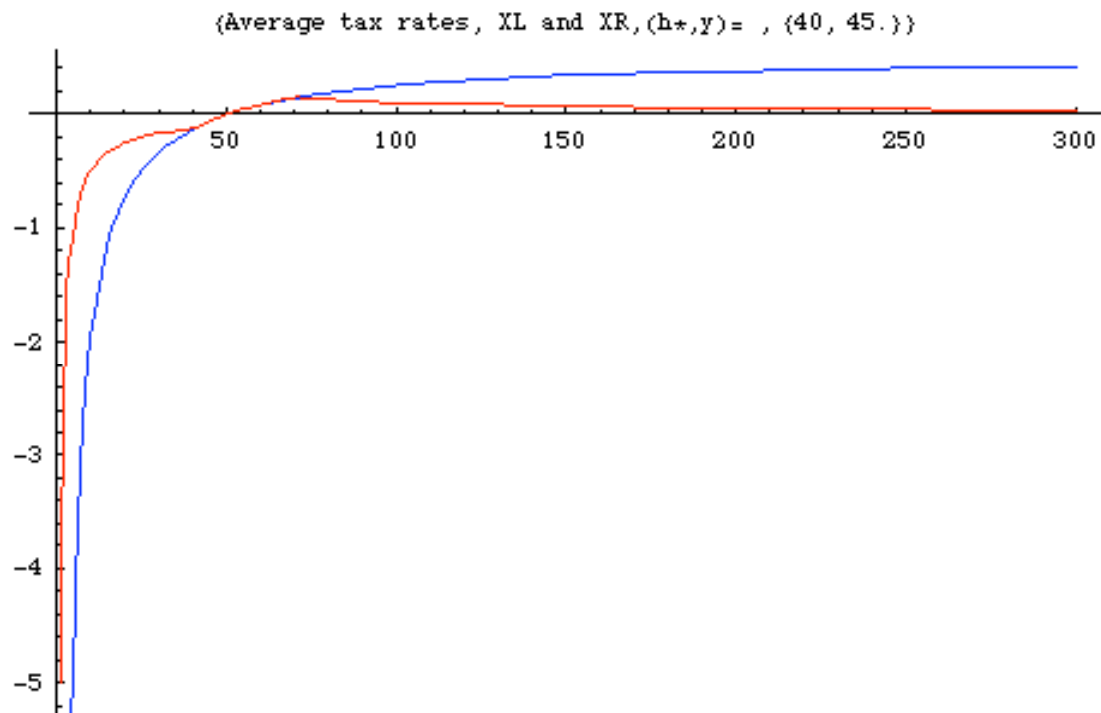


Figure 4 The average tax rate functions for a pair of equilibrium policies



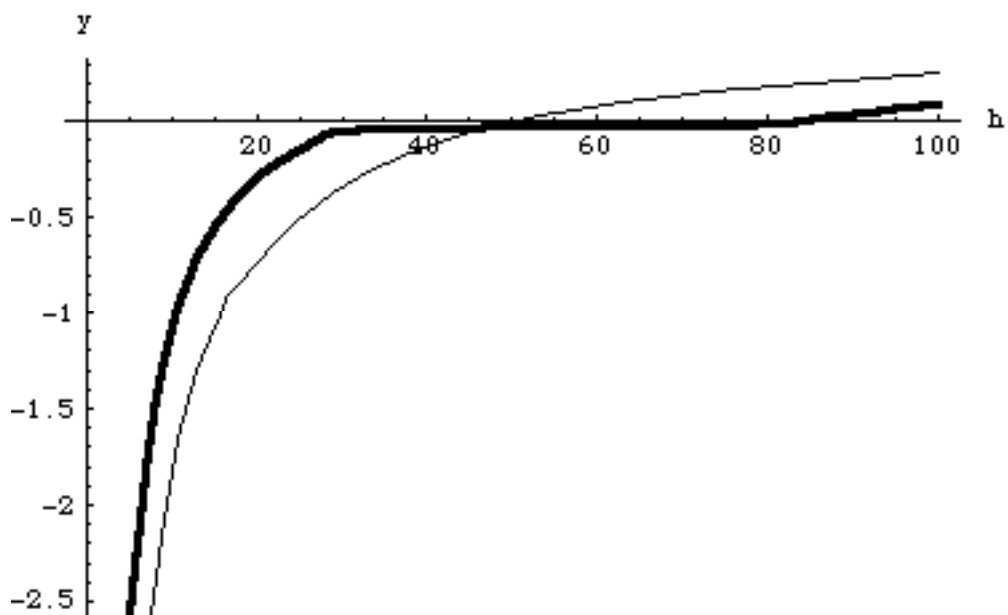


Figure 5a. Average tax rates, Left policy, at  $h_* = 20$  (plain) and  $h_* = 80$  (bold)

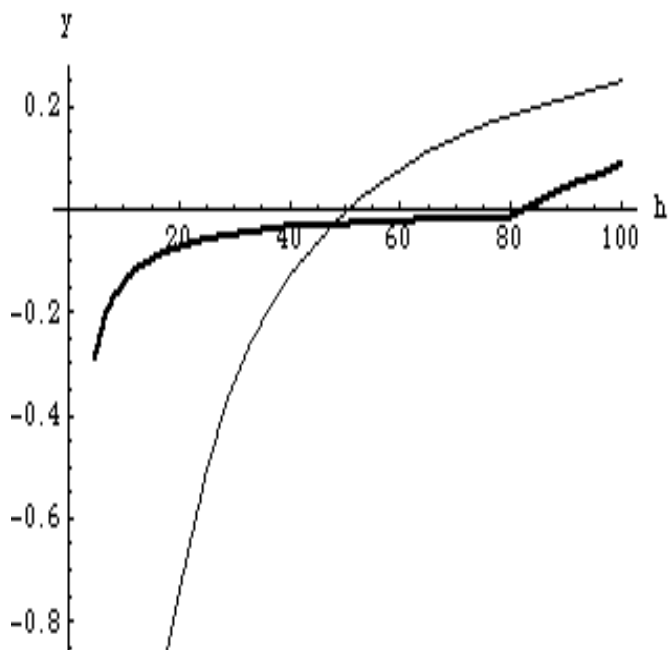


Figure 5b. Average tax rates, Right policy, at  $h_* = 20$  (plain) and  $h_* = 80$  (bold)

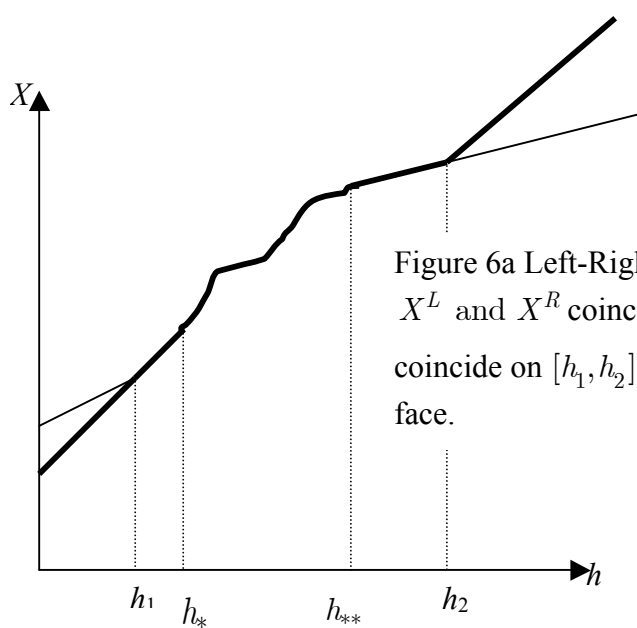
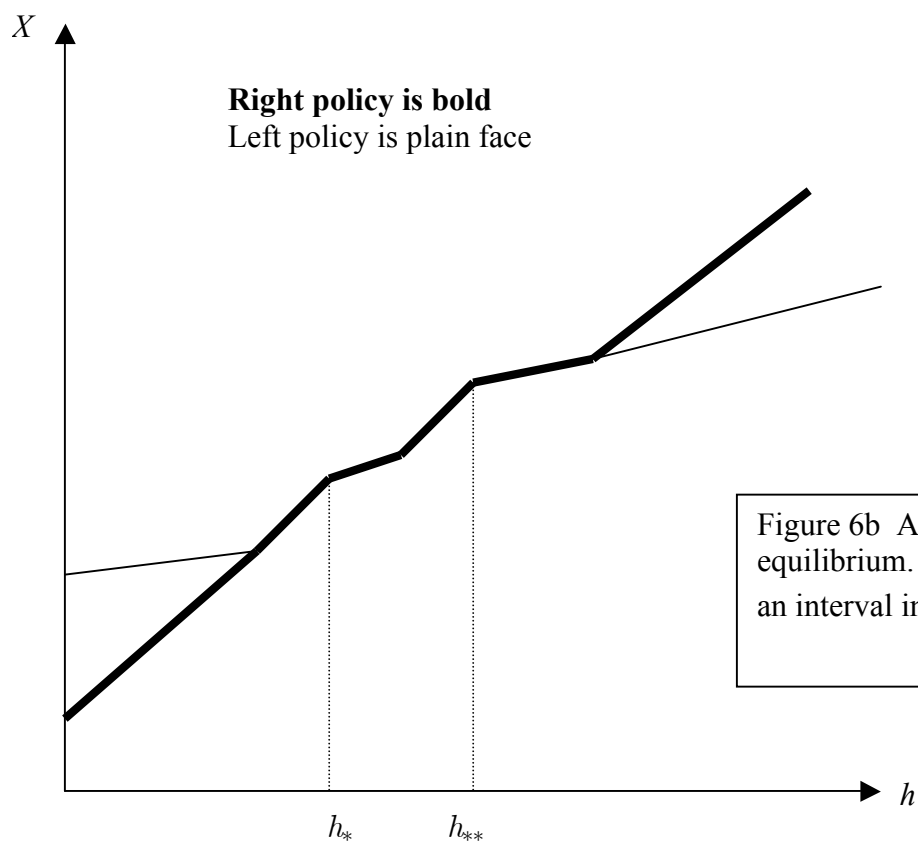


Figure 6a Left-Right stationary equilibrium where  $X^L$  and  $X^R$  coincide with arbitrary  $Z$  on  $[h_*, h_{**}]$  and coincide on  $[h_1, h_2]$ .  $R$  policy in bold,  $L$  policy in plain face.



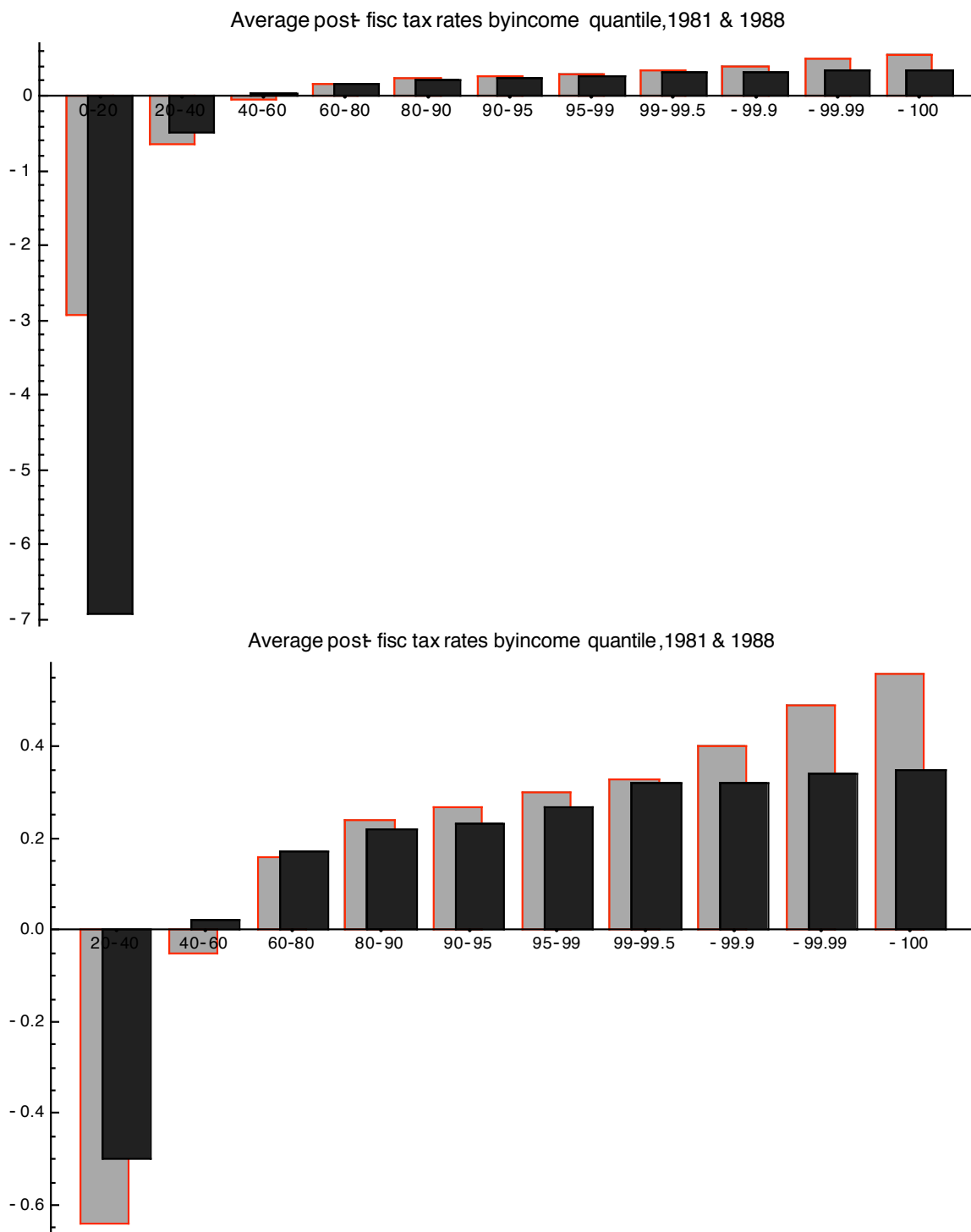


Figure 7a. Light bars are 1981 (Carter); dark bars are 1988 (Reagan). Lower panel excises the bottom quintile and re-scales.

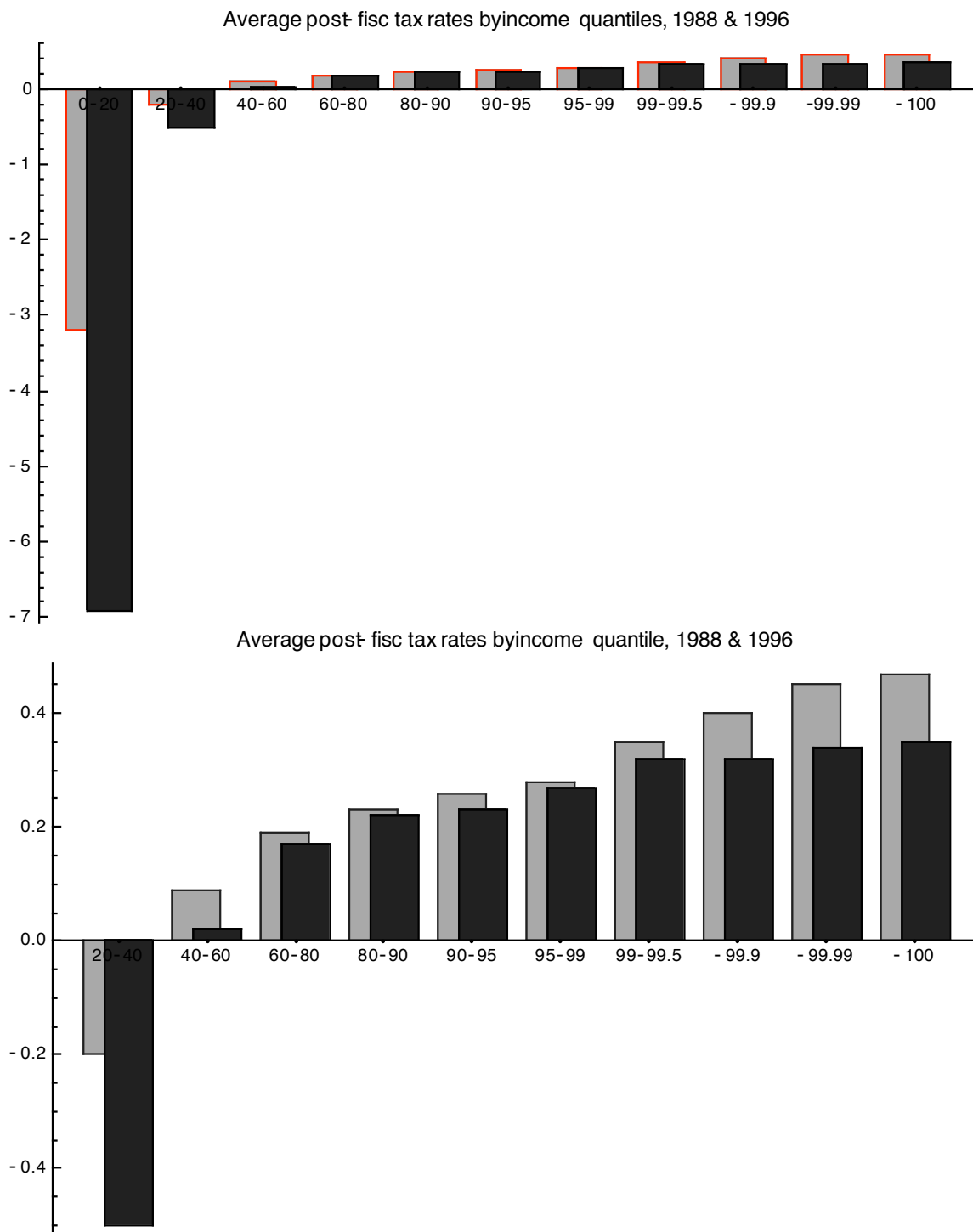


Figure 7b. Dark bars are 1988 (Reagan), light bars are 1996 (Clinton). Lower panel excises bottom quintile and re-scales.

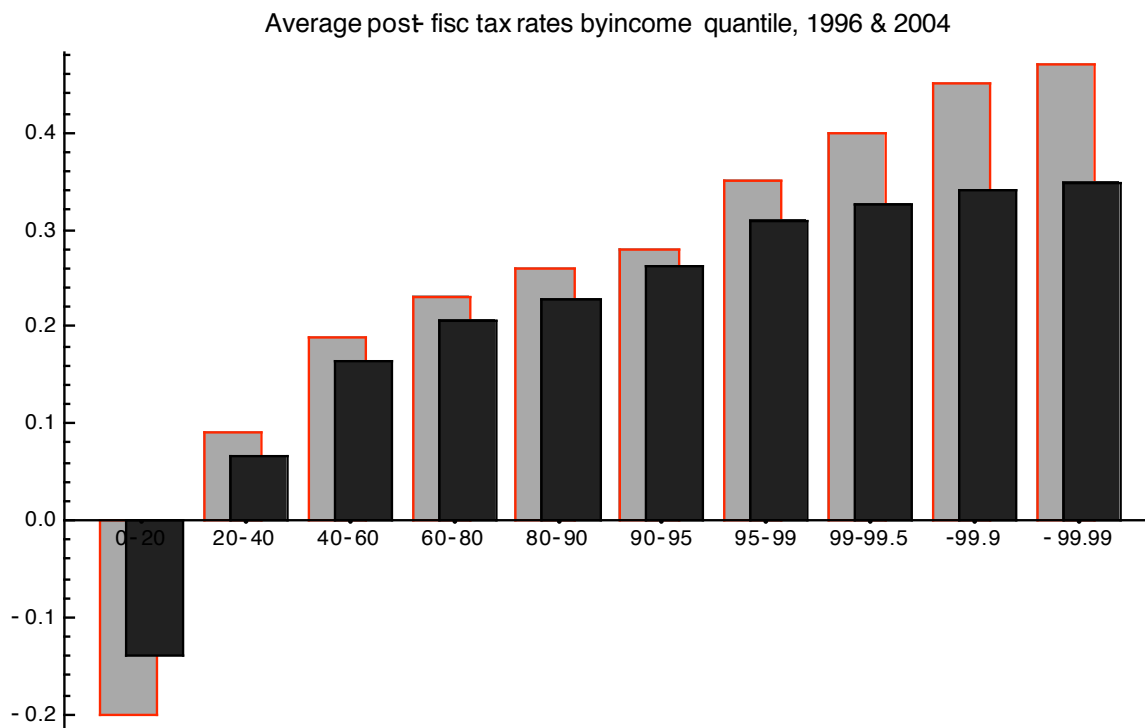


Figure 7c Dark bars, 2004 (G.W. Bush), light bars, 1996 (Clinton).

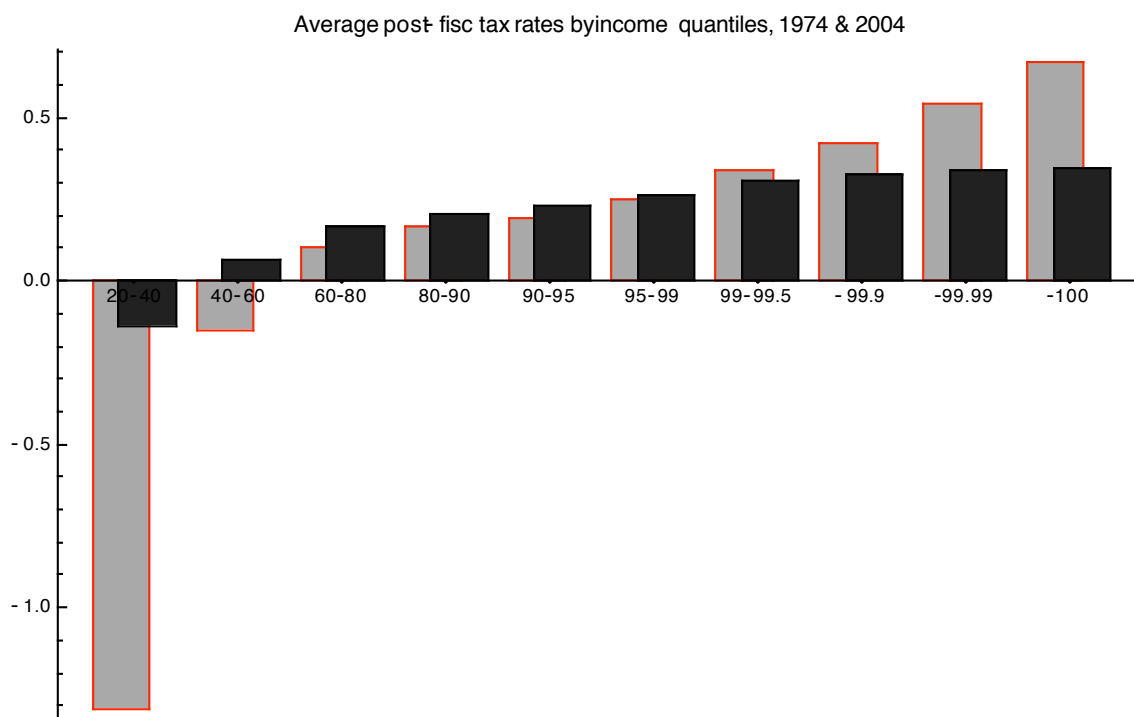
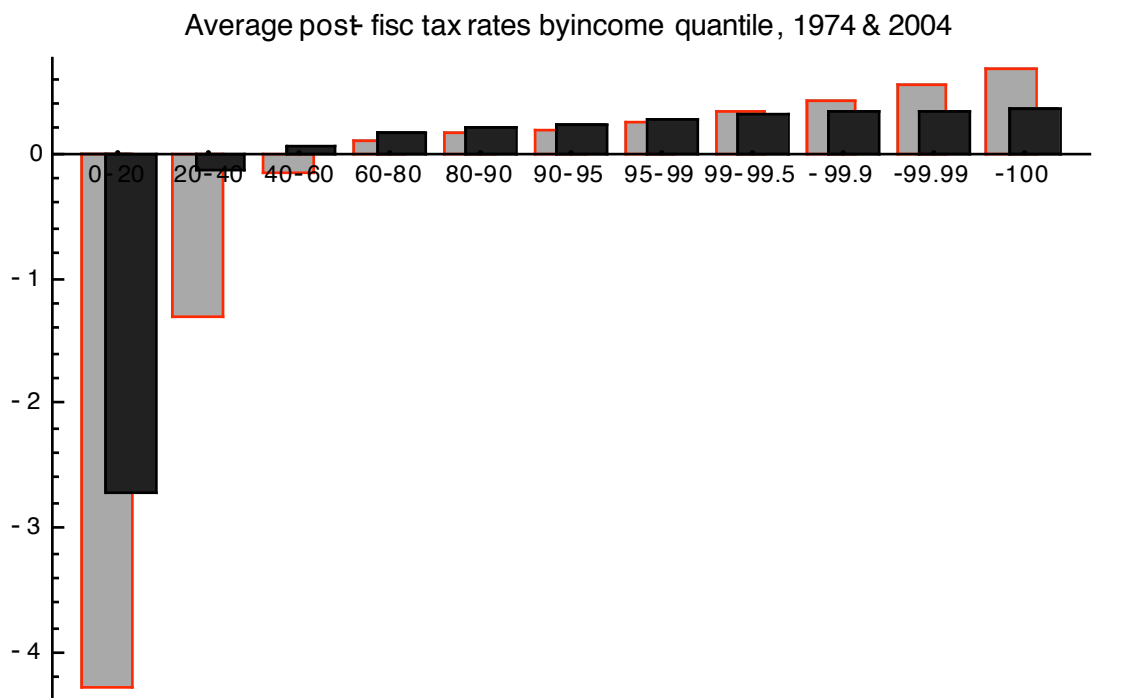


Figure 7d. Light bars, 1974; dark bars, 2004.

Figure 8 The empirical function  $\theta$ : Democratic presidential candidate vote share, by voter income quantile, various years

