Boolean Gröbner bases and Sudoku

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For a boolean ring $\mathbf{B}$, a residue class ring $\mathbf{B}[X_1, \ldots, X_n]/\langle X_1^2 + X_1, \ldots, X_n^2 + X_n \rangle$ (denoted by $\mathbf{B}(X_1, \ldots, X_n)$) is called a *boolean polynomial ring.* Gröbner bases in such a boolean polynomial ring (*boolean Gröbner bases*) are first introduced in [5] and further developments are done in [1],[3] and [4]. (Note that $\mathbf{B}$ can be an arbitrary computable boolean ring, which of course is not restricted to the Galois field $GF(2)$ which is the simplest instance of a boolean ring.)

A Sudoku puzzle can be represented as a system of polynomial equations of a boolean polynomial ring $\mathbf{B}(A_{(1,1)}, \ldots, A_{(9,9)})$, where $\mathbf{B}$ is the boolean ring of the power set of $\{1, 2, \ldots, 9\}$ i.e. the set of all subsets of $\{1, 2, \ldots, 9\}$. Each variable $A_{(i,j)}(i, j = 1, \ldots, 9)$. represents the cell on the ith row and the jth column. Constraints of Sudoku regulation are transformed into polynomial equations. For example, the constraint " The cells of the ith raw contain different numbers. " is represented by the polynomial equations $A_{(i,j)}A_{(i,k)} = 0(1 \leq j < k \leq 9)$ and $A_{(i,1)} + \cdots + A_{(i,9)} = \{1, 2, \ldots, 9\}$.

We show that Sudoku puzzles can be solved by the computations of boolean Gröbner bases quite effectively. Most Sudoku puzzles are momentarily solved even by our old implementation of boolean Gröbner bases of [2].

References

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